PROXIMITIES AND EXTENSIONS OF CONTINUOUS FUNCTIONS

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BY
MANI GAGRAT

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CERTIFICATE

This is to certify that the thesis entitled "Proximities and Extensions of Continuous Functions" by Miss Mani Gagrat being submitted for the award of the Degree of Doctor of Philosophy of the Indian Institute of Technology, Kanpur, is a record of bonafide research work carried out by her under my supervision. The thesis has, in my opinion, reached the standard fulfilling the requirements for the Ph.D. degree. The results embodied in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

S A Naimpally

21st March, 1970.

(S.A. Naimpally)
Supervisor

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CONTENTS

ABSTRACT

PROXIMITIES AND EXTENSIONS OF CONTINUOUS FUNCTIONS

Theorems dealing with the continuous extensions of functions from dense subspaces have always constituted a very important class of problems in Topology. Closely related to these are the compactification theorems. In the present dissertation, we shall obtain a unified approach, based on the concept of proximities defined on sets, to solve both these types of problems.

Let X be a non-void set and let δ be a binary relation defined on the power set of X consider the following axioms :

(P.1) $A \delta B$ implies $B \delta A$.

(P.2) $(A \cup B) \delta C$ iff $A \delta C$ or $B \delta C$.

(P.3) $A \delta B$ implies that $A \neq \phi$, $B \neq \phi$.

(P.4) $A \cap B \neq \phi$ implies $A \delta B$.

(P.5) $x \delta y$ implies $x = y$.

(P.6) $A \not\delta B$ implies there exists an $E \subset X$ such that $A \not\delta E$ and $(X-E) \not\delta B$.

(P.6') $A \delta B$, $b \delta C$ for every b in B implies that $A \delta C$.

(P.6") $a \delta B$, $b \delta C$ for every b in B implies that $a \delta C$.

Then (X,δ) is an Efremovič (or an EF-)space if δ satisfies (P.1), (P.2), (P.3), (P.4) and (P.6). (X,δ) is a Lodato (or a LO-) space if δ satisfies (P.1), (P.2), (P.3), (P.4) and (P.6'). Finally, (X,δ) is a separation [or an S-]space if δ satisfies (P.1), (P.2), (P.3), (P.4), (P.5) and (P.6").

The first chapter gives the basic definitions and theorems, which will be frequently used in the subsequent work. Most of these are known results, and care is being taken to provide adequate references for them.

Chapter 2 contains the fundamental theorems on extensions of functions. The main result in this direction is that any proximally continuous map $f : (X,\delta_1) \to (Y,\delta_2)$, where (X,δ_1) and (Y,δ_2) are separated LO-spaces, has a continuous extension $f_\Sigma : \Sigma_X \to \Sigma_Y$, where Σ_Z denotes the space of all bunches over the LO-space (Z,δ) with the Absorption topology. Using these results, we derive, for LO-spaces, a satisfactory generalization of the well-known Smirnov compactification theorem for EF-spaces. This theorem reads as follows :-

Let (X,δ) be a separated LO-space such that if $A \,\delta\, B$ then there is a bunch over (X,δ) containing both A and B. Then :

(i) there exists a compact T_1-space $\underset{\sim}{X}$ (the space of all maximal bunches over (X,δ) with the Absorption topology) containing a dense homeomorphic copy of X ;

(ii) $A \,\delta\, B$ iff $Cl(\Phi(A)) \cap Cl(\Phi(B)) \neq \phi$ in $\underset{\sim}{X}$ (Φ being the embedding map of X into $\underset{\sim}{X}$).

(iii) If $f : (X,\delta) \to (Y,\delta')$ is proximally continuous (where (Y,δ') is a separated LO-space), then f has a continuous extension $\overline{f} : \underset{\sim}{X} \to \Sigma_Y$.

In the third chapter, a general result on extensions of continuous functions is first proved. Let X be a T_1 dense subspace of an Ro-space T and let (Y,δ) be an Efremovič proximity space.

Then a continuous function $f : X \to Y$ has a continuous extension $\overline{f} : T \to Y$ (the Smirnov compactification of Y) if and only if $A \not\delta B$ in Y implies $Cl_T f^{-1}(A) \cap Cl_T f^{-1}(B) = \phi$. It is then shown that the extension results of Taimanov, McDowell, Blefko and Engelking are easy consequences of this theorem. Not only these, but we also find new extension theorems which extend Taimanov's result to cases when the range space of the continuous map is some generalization of compact spaces. Thus, we prove that if X is a T_1-dense subspace of an Ro-space T_1 and if Y is a locally compact Hausdorff space, then a continuous map $f : X \to Y$ has a continuous extension $\overline{f} : T \to Y$ iff (i) for disjoint closed sets F_1, F_2 in Y, $Cl_T f^{-1}(F_1) \cap Cl_T f^{-1}(F_2) = \phi$ and (ii) for each t in T, the family $f_\Sigma(\sigma^t) = \{A \in P(Y) : t \in Cl_T f^{-1}(A^-)\}$ contains some compact subset C_t of Y.

The fourth chapter is devoted to the various compactifications of topological spaces. In the first half of this section, we exploit the known results of proximity spaces to show that a separating [normal] base $\mathcal{L}$ on a T_1-[Tychonoff] space X induces a LO-[respectively EF-] proximity $\delta(\mathcal{L})$ on X and that the Wallman compactification $W(\mathcal{L})$ is homeomorphic to the Smirnov compactification of X corresponding to $\delta(\mathcal{L})$. We then use this result to derive a necessary and sufficient condition for a Hausdorff compactification to be Wallman. In the latter portion of this chapter, we utilize the results of chapter 2 to deduce the well-known theorem of Ponomarev./

Chapter 5 deals with real compact spaces and Wallman real-compactifications. Extension theorems involving real-compact spaces will be derived by the help of our earlier results. A concrete

realization of the Hewitt real compactification νX of a Tychonoff space X will also be given. Next, it will be shown that the space $\eta(\mathcal{L})$, corresponding to a countably productive normal base $\mathcal{L}$ on a space X, is real compact iff $\bigcap_{n=1}^{\infty} \mathrm{Cl}_{X^0}(L_n) = \mathrm{Cl}_{X^0}(\bigcap_{n=1} L_n)$, $L_n \in \mathcal{L}$, where X^0 denotes the 0-closure of X in $W(\mathcal{L})$. Finally, necessary and sufficient conditions shall be obtained for $\eta(\mathcal{L})$ to be homeomorphic to an $\mathcal{L}^*$-real compactification Y of X. These results will be motivated by the known results concerning Wallman compactifications.

In chapter 6, we will make a detailed study of S-spaces Analogies will be developed between these spaces and the well-known EF-spaces and the LO-spaces. For instance, it will be proved that every S-space has a compatible T_1-topology and further, that it also has a compatible generalized uniform structure.

In the concluding chapter, we shall consider the problem of proximal embedding of S-spaces. In particular, it will be shown that every abstract S-proximity is a subspace S-proximity of δ_0' defined by $A \, \delta_0' \, B$ iff $(A \cap \overline{B}) \cup (\overline{A} \cap B) \neq \phi$. The generalization of Smirnov Theorem for S-spaces will also be deduced, and finally, by extending our previous results of LO-spaces to the S-spaces, further variations of Taimanov's result will be obtained.

CHAPTER 1

PRELIMINARIES

In this chapter, our objective is to familiarize the reader with the basic theory of proximity spaces, and at the same time, to recall some of the known definitions and results which will be relevant in the subsequent work. As most of the results mentioned in this section have already appeared in print elsewhere, we shall be omitting their proofs. The terminology followed in the entire course of this dissertation is that of Pervin [28].

We shall denote the power set of a non-void set X by $P(X)$. Let δ be a binary relation defined on $P(X)$. For $(A,B) \in \delta$, we will write $A \,\delta\, B$, and for $(A,B) \notin \delta$, we will write $A \not\delta B$. For $x \in X$, $\{x\} \in P(X)$ will often be written as x. Consider the following set of axioms:

(P.1) $A \,\delta\, B$ implies $B \,\delta\, A$.

(P.2) $(A \cup B) \,\delta\, C$ iff $A \,\delta\, C$ or $B \,\delta\, C$.

(P.3) $A \,\delta\, B$ implies that $A \neq \phi$, $B \neq \phi$.

(P.4) $A \cap B \neq \phi$ implies $A \,\delta\, B$.

(P.5) $x \,\delta\, y$ implies $x = y$.

(P.6) $A \not\delta B$ implies there exists an $E \in P(X)$ such that $A \not\delta E$ and $(X-E) \not\delta B$.

(P.6') $A \,\delta\, B$, $b \,\delta\, C$ for every b in B implies that $A \,\delta\, C$.

(P.6") $a \,\delta\, B$, $b \,\delta\, C$ for every b in B implies that $a \,\delta\, C$.

1.1 Definition : (Efremovič [8]) δ is a [separated] Efremovič proximity (denoted by [separated] EF-proximity) on X iff δ satisfies the axioms (P.1), (P.2), (P.3), (P.4), [(P.5)] and (P.6).

1.2 Definition : (Lodato [18]) δ is a [separated] Lodato proximity (denoted by [separated] LO-proximity) on X if δ satisfies the axioms (P.1), (P.2), (P.3), (P.4), [(P.5)] and (P.6').

1.3 Definition : (Krishna Murti [24], Szymanski [36], Wallace [38,39] for a survey see Pervin [29]). δ is a separation proximity (denoted by S-proximity) iff δ satisfies the axioms (P.1), (P.2), (P.3), (P.4), (P.5) and (P.6'').

1.4 Remark : The axiom (P.6) is often called the Strong Axiom. It has been proved (see Naimpally and Warrack [26]) that the following axiom (P_1.6) though apparently stronger than the Strong Axiom, is actually equivalent to it:

(P_1.6) $A \not\delta B$ implies the existence of E,F in P(X) such that $A \not\delta (X-E)$, $B \not\delta (X-F)$ and $E \not\delta F$. As a corollary to this, we observe that in an EF-space (X,δ), if $A \not\delta B$, then A, B are contained in neighbourhoods (nbhds.) N_A, N_B respectively such that $N_A \not\delta N_B$.

1.5 Remark : The pair (X,δ) will be called an EF-space, a LO-space or an S-space depending on whether δ is an EF-proximity, a LO-proximity, or an S-proximity respectively. In general, "proximity δ " will stand for any one of the above mentioned proximities, and similarly, a "p-space (X,δ)" will be any of the above listed spaces, unless otherwise specified.

1.6 Proposition : Every EF-space is a LO-space, and every separated LO-space is an S-space. The converse need not hold for either of these.

1.7 Proposition : (Efremovič [8], [Lodato [20]) Every EF-[LO-] proximity δ on a set X induces a topology $\tau(\delta)$ on X as follows :

For G in $P(X)$, $G \in \tau(\delta)$ iff for each $x \in G$, $x \not\delta (X-G)$.

1.8 Definition : If τ is a topology on a p-space (X,δ) such that $\tau = \tau(\delta)$ then τ and δ are said to be compatible with each other.

1.9 Definition : A topological space (X,τ) is R_o iff either of the following equivalent conditions is satisfied.

(i) $x \in G \in \tau$ implies $\overline{x} \subset G$.

(ii) $x \in \overline{y}$ implies that $y \in \overline{x}$.

1.10 Theorem : (Lodato [18]). If δ is a [separated] LO-proximity on a set X, then $(X,\tau(\delta))$ is R_o $[T_1]$.Conversely, every R_o-$[T_1$-$]$ space (X,τ) has a compatible [respectively separated] LO-proximity δ_o defined by :

For A,B in $P(X)$, $A \; \delta_o \; B$ iff $A^- \cap B^- \neq \phi$.

1.11 Theorem : (Efremovič [8]). If δ is a [separated] EF-proximity on a set X, then $(X,\tau(\delta))$ is completely regular [respectively Tychonoff]. Conversely, every completely regular [Tychonoff] space (X,τ) has a compatible [respectively separated] EF-proximity δ_F defined by :

For A,B in $P(X)$, $A \not\delta_F B$ iff there exists a continuous map $f : X \to [0,1]$ such that $f(A) = 0$ and $f(B) = 1$.

(The above EF-proximity δ_F is usually called the functionally distinguishable proximity.)

1.12 Remark : δ_o, as defined in 1.10, is a compatible EF-proximity on (X,τ) iff τ is T_4. Further, for a compact Hausdorff space (X,τ), δ_o is the unique compatible EF-proximity.

1.13 Proposition : (Efremovič [8], [Lodato [18]]). In an EF-[LO-] space (X,δ), $A \; \delta \; B$ iff $A^- \; \delta \; B^-$.

1.14 Proposition : In a p-space (X,δ), $A \delta B$, $A \subset E$, $B \subset F$ implies that $E \delta F$.

Proof. As $E = A \cup (E - A)$, $A \delta B$ implies, from (P.2) that $E \delta B$. Similarly, $F = B \cup (F - B)$, $E \delta B$ implies that $E \delta F$.

1.15 Definition : If δ_1, δ_2 are two proximities on a set X, then we define $\delta_1 > \delta_2$ iff $A \delta_1 B$ implies that $A \delta_2 B$. This is also expressed by saying that δ_1 is finer than δ_2 or that δ_2 is coarser than δ_1.

1.16 Theorem (Smirnov [31], [Lodato [18]]) δ_F $[\delta_o]$ as defined in 1.11 [respectively 1.10] is the finest EF-[respectively LO-] proximity compatible with any completely regular [respectively R_o-] topological space (X,τ).

1.17 Remark : It is worthwhile to note here that every metric space (X,d) has a compatible EF-proximity δ_M, called the metric proximity given by :

$$A \,\delta_M\, B \text{ iff } D(A,B) = 0 \text{ , where } D(A,B) = \inf_{a\in A,\, b\in B} d(a,b).$$

1.18 Definition : If (X,δ_1), (Y,δ_2) are two p-spaces, then a map $f : (X,\delta_1) \to (Y,\delta_2)$ is said to be proximally continuous (or p-continuous) iff $A \delta_1 B$ implies that $f(A) \delta_2 f(B)$. Further, f is a proximal isomorphism between (X,δ_1) and (Y,δ_2) iff f is one-to-one, onto, and such that both f and f^{-1} are p-continuous.

1.19 Proposition : Let (X,δ_1) be an EF-space, and let (Y,δ_2) be any p-space. If $f : (X,\delta_1) \to (Y,\delta_2)$ is p-continuous, then the map $f : (X,\tau(\delta_1)) \to (Y,\tau(\delta_2))$ is continuous. In general the converse need

not hold; but it holds if $\delta_1 = \delta_F$ and δ_2 is any EF-proximity on Y.

1.20 Proposition : Let (X,δ_1) be a LO-space and let (Y,δ_2) be any p-space. If $f : (X,\delta_1) \to (Y,\delta_2)$ is p-continuous, then the map $f : (X,\tau(\delta_1)) \to (Y,\tau(\delta_2))$ is continuous. In general, the converse need not hold; but it holds if δ_1 = LO-proximity δ_o and δ_2 is any LO-proximity on Y.

Proof : Suppose f is p-continuous. To show that f is continuous, we need prove that if $x \in A^-$, then $f(x) \in \overline{f(A)}$ for each $A \in P(X)$. Let $x \in A^-$. Then $x\ \delta_1\ A$ so that $f(x)\ \delta_2\ f(A)$. Hence $f(x) \in \overline{f(A)}$. That the converse is not always true can be seen by taking $X = Y = \mathbb{R}$, $\delta_1 = \delta_M$ and δ_2 = the LO-proximity δ_o. Then the identity map i is continuous, but not p-continuous. Finally, suppose f is continuous and $\delta_1 = \delta_o$. If $A\ \delta_1\ B$, then $A^- \cap B^- \neq \phi$, so that $f(\overline{A}) \cap f(\overline{B}) \neq \phi$. Since f is continuous, $f(\overline{A}) \subset \overline{f(A)}$ and $f(\overline{B}) \subset \overline{f(B)}$. Hence $\overline{f(A)} \cap \overline{f(B)} \neq \phi$, which, in view of (P.4), implies that $\overline{f(A)}\ \delta_2\ \overline{f(B)}$. Using 1.13, we get that $f(A)\ \delta_2\ f(B)$.

1.21 Definition : If (X,δ) is a p-space, and $Y \subset X$, then noting that $P(Y) \subset P(X)$, we define a binary relation δ_Y on $P(Y)$ as :

For $A,B \in P(Y)$, $A\ \delta_Y\ B$ iff $A\ \delta\ B$.

1.22 Remark : The relation δ_Y so defined on $P(Y)$ is a proximity on Y, called the induced subspace proximity. Further, $\tau(\delta_Y)$ is the same as the subspace topology induced on Y by $\tau(\delta)$.

1.23 Definition : A non-void class σ of subsets of a p-space (X,δ) is a bunch [cluster] over (X,δ) iff σ satisfies the following set of axioms:

(B.1) $A\ \delta\ B$ for every A,B in σ.

(B.2) $(A \cup B) \in \sigma$ iff $A \in \sigma$ or $B \in \sigma$.

(B.3) $A \in \sigma$ iff $A^- \in \sigma$.

[respectively (B.3') $A \delta B$ for all B in σ implies $A \in \sigma$].
Further, a bunch σ over (X,δ) is called a maximal bunch iff $\sigma \subset \sigma_1$, σ_1 a bunch over (X,δ), implies that $\sigma = \sigma_1$.

1.24 Remark : Every cluster is a bunch ; in fact, it is a maximal bunch. However, not every bunch is a cluster. For example, the family $\{A \in P(X) : A \delta x, A \neq \{x\}\}$ is a bunch over any non-discrete LO-space (X,δ), but it is not a maximal bunch, and hence neither a cluster.

1.25 Proposition : Every bunch over a LO-space (X,δ) is contained in a maximal bunch.

Proof : Let σ_1 be a bunch over a LO-space (X,δ) and let $\underset{\sim}{G}$ be the family of all bunches over (X,δ) which contain σ_1. That $\underset{\sim}{G}$ is non-void follows from the fact that $\sigma_1 \in \underset{\sim}{G}$. Partially order $\underset{\sim}{G}$ by inclusion. Then every chain $\underset{\sim}{F}$ in $\underset{\sim}{G}$ clearly has the union of its members as an upper bound. Hence, by Zorn's lemma, $\underset{\sim}{G}$ has a maximal element, which is obviously the required maximal bunch containing σ_1.

1.26 Proposition : (Lodato [18,19]). In a LO-space (X,δ), the following hold :

(1) If σ is a bunch, $A \in \sigma$ and $A \subset B$, then $B \in \sigma$. Hence, in particular, $X \in \sigma$.

(2) For $A \in P(X)$, and σ a bunch over (X,δ), either $A \in \sigma$ or $(X-A) \in \sigma$.

(3) For $x \in X$, $\sigma_x = \{A \in P(X) : x \delta A\}$ is a cluster over X, called the point cluster.

(4) If σ is a bunch and $\{x\} \in \sigma$, then $\sigma = \sigma_x$.

1.27 Definition : Let X be a set and $\mathfrak{X}$ some distinguished ring (i.e. closed under finite unions and finite intersections) of subsets of X

which contains X. If $\underset{\sim}{F}$ is a non-void subfamily of $\mathcal{L}$, then :

(a) $\underset{\sim}{F}$ is an <u>$\mathcal{L}$-filter</u> iff

(i) $\underset{\sim}{F}$ is closed under finite intersections.

(ii) $\underset{\sim}{F}$ contains every superset in $\mathcal{L}$ of each of its members.

(iii) $\phi \notin \underset{\sim}{F}$.

(b) $\underset{\sim}{F}$ is a <u>prime $\mathcal{L}$-filter</u> iff $\underset{\sim}{F}$ is an $\mathcal{L}$-filter such that if $L_1, L_2 \in \mathcal{L}$, and $(L_1 \cup L_2) \in \underset{\sim}{F}$, then either $L_1 \in \underset{\sim}{F}$ or $L_2 \in \underset{\sim}{F}$.

(c) $\underset{\sim}{F}$ is an <u>$\mathcal{L}$-ultrafilter</u> iff $\underset{\sim}{F}$ is an $\mathcal{L}$-filter which is maximal, i.e. if $L \in \mathcal{L}$ and $L \cap F \neq \phi$ for every F in $\underset{\sim}{F}$, then $L \in \underset{\sim}{F}$.

Note that if $\mathcal{L} = P(X)$, we obtain the filters, prime-filters and ultrafilters in the usual sense.

(d) $\underset{\sim}{F}$ is a <u>real $\mathcal{L}$-filter</u> iff $\underset{\sim}{F}$ is an $\mathcal{L}$-filter which has the countable intersection property (or c.i.p.).

1.28 <u>Proposition</u> : Let (X,δ) be a LO-space, and let $\mathcal{L}$ be a ring of closed subsets in $(X,\tau(\delta))$. If $\underset{\sim}{F}$ is an $\mathcal{L}$-ultrafilter on X, then $b(\underset{\sim}{F}) = \{A \in P(X) : A^- \in \underset{\sim}{F}\}$ is a bunch over (X,δ), called the <u>bunch generated by</u> $\underset{\sim}{F}$.

<u>Proof</u> : (i) $A, B \in b(\underset{\sim}{F})$ implies that $\overline{A}, \overline{B} \in \underset{\sim}{F}$ which in turn implies that $\overline{A} \cap \overline{B} \neq \phi$. Thus $\overline{A}\ \delta\ \overline{B}$, and hence, by Prop. 1.13, $A\ \delta\ B$, verifying (B.1).

(ii) $(A \cup B) \in b(\underset{\sim}{F})$ iff $(A \cup B)^- \in \underset{\sim}{F}$ iff $\overline{A} \cup \overline{B} \in \underset{\sim}{F}$ iff $\overline{A} \in \underset{\sim}{F}$ or $\overline{B} \in \underset{\sim}{F}$ iff $A \in b(\underset{\sim}{F})$ or $B \in b(\underset{\sim}{F})$. Hence (B.2) holds.

(iii) (B.3) follows trivially as $A \in b(\underset{\sim}{F})$ iff $\overline{A} \in \underset{\sim}{F}$ iff $\overline{A} \in b(\underset{\sim}{F})$ [as $\underset{\sim}{F} \subset b(\underset{\sim}{F})$].

1.29 <u>Proposition</u> : (Mrowka [22]). In an EF-space (X,δ), a non-void family σ of subsets of X is a cluster iff there exists an ultrafilter $\underset{\sim}{F}$ in X such that $\sigma = \sigma(\underset{\sim}{F}) = \{A \in P(X) : A \,\delta\, F$ for every F in $\underset{\sim}{F}\}$. (the cluster generated by $\underset{\sim}{F}$).

1.30 <u>Definition</u> : A non-void subset σ of $P(X)$ in a topological space (X,τ) is said to <u>converge</u> to a point x in X iff the nbhd. filter of x is a subclass of σ .

1.31 <u>Definition</u> : Let (X,τ) be a T_1-space, and let $\mathcal{L}$ be an arbitrary ring of closed subsets of X, then :

(a) $\mathcal{L}$ is <u>separating</u> iff $x \notin F$, F closed in X, implies the existence of L_1, L_2 in $\mathcal{L}$ such that $x \in L_1$, $F \subset L_2$ and $L_1 \cap L_2 = \phi$.

(b) $\mathcal{L}$ is <u>disjunctive</u> iff $x \notin F$, F closed in X, implies the existence of an L in $\mathcal{L}$ such that $x \in L \subset (X-F)$.

(c) $\mathcal{L}$ is a <u>normal base</u> on X iff $\mathcal{L}$ is a disjunctive base for closed sets in X satisfying the following axiom :

(N) : $L_1, L_2 \in \mathcal{L}$, $L_1 \cap L_2 = \phi$ implies the existence of L_1', L_2' in $\mathcal{L}$ such that $L_1 \subset X-L_1'$, $L_2 \subset X-L_2'$, and $L_1' \cup L_2' = X$.

1.32 <u>Remark</u> : (1) We note that a disjunctive base of closed sets is always separating; for $x \notin F$, F closed in X, together with the fact that $\mathcal{L}$ is a base for closed sets in X implies the existence of an L_1 in $\mathcal{L}$ such that $x \in X-L_1 \subset X-F$, and hence, by 1.31(b), there exists an L_2 in $\mathcal{L}$ such that $x \in L_2 \subset X-L_1 \subset X-F$. Clearly, $L_1 \cap L_2 = \phi$ i.e. $\mathcal{L}$ is separating.

(2) We also observe here that if $\mathcal{L}$ is a separating base on a T_1-space (X,τ), then ϕ and X both belong to $\mathcal{L}$. That $X \in \mathcal{L}$ is a direct outcome of the fact that X is closed and $\mathcal{L}$ is a base for closed sets in X.

To see that $\phi \in \mathcal{L}$, we consider the following cases : (a) If $X = \{x\}$, then as X, ϕ are both closed; and hence, as $\mathcal{L}$ is a base for closed sets, $\mathcal{L} = \{\{x\}, \phi\}$ (b) If $x,y \in X$, $x \neq y$. Then $x \notin \overline{y} = y$ and hence, by 1.31 (a), there exist L_x, L_y in $\mathcal{L}$ such that $x \in L_x$, $y \in L_y$ and $L_x \cap L_y = \phi$. Since $\mathcal{L}$ is a ring, $L_x \cap L_y = \phi \in \mathcal{L}$.

1.33 Definition : A non-void family $\mathcal{L}$ of subsets of X is called [countably] productive iff $\mathcal{L}$ is closed under finite [respectively countable] intersections of its members.

1.34 Theorem : (Frink [11]). A topological space (X,τ) is Tychonoff iff it has a normal base.

1.35 Definition: (see Wallman [40], Steiner [35]). Let (X,τ) be a topological space, and let $\mathcal{L}$ be a distinguished family of closed sets in X. Define $W(X,\mathcal{L})$ to be the set of all $\mathcal{L}$-ultrafilters on X. For $L \in \mathcal{L}$, let $L^* = \{\underset{\sim}{F} \in W(X,\mathcal{L}) : L \in \underset{\sim}{F}\}$. Then the Wallman Space of $(X,\mathcal{L})$ is the space $W(X,\mathcal{L})$, with the Wallman topology (or the W-topology) induced on X by taking the family $\{L^* : L \in \mathcal{L}\}$ as a subbase for closed sets in $W(X,\mathcal{L})$.

1.36 Remark : Frink [11] has proved that if (X,τ) is Tychonoff and if $\mathcal{L}$ is a normal base on X, then the Wallman space $W(X,\mathcal{L})$ is, in fact, a compactification of X. Steiner [35] has proved the same result when (X,τ) is T_1 and $\mathcal{L}$ is a separating base for closed sets in (X,τ). In either case, the homeomorphism $w : X \to w(X,\mathcal{L})$, defined by $w(x) = \{L \in \mathcal{L} : x \in L\}$ for $x \in X$, is called the Wallman map.

1.37 Definition : A subset E of a topological space (X,τ) is a zero-set iff there exists a continuous map $F : X \to \mathbb{R}$ such that $f^{-1}(0) = E$.

The zero-set E of the function f is denoted by $Z(f)$, and the family of all zero-sets of X is denoted by $Z(X)$, or simply by Z when no confusion is possible.

1.38 Proposition : (Gillman and Jerison [12]) For a topological space (X,τ), the following hold :

(i) Z is closed under countable intersections of its members.

(ii) Two sets in (X,τ) are functionally separated iff they are contained in disjoint zero-sets.

(iii) Every nbhd. of a point in a completely regular space contains a zero-set nbhd. of the point.

(iv) If (X,τ) is normal, then every closed G_δ is a zero-set.

1.39 Lemma : (Frink [11]) In a Tychonoff space X, Z is a normal base.

Let X be a dense subspace of a topological space T, and let $\mathcal{L}$ be a ring of closed sets in X. If $\underset{\sim}{F}$ is an $\mathcal{L}$-filter on X, then :

1.40 Definition : $\underset{\sim}{F}$ is said to cluster at a point p in T iff

$$p \in \bigcap_{F \in \underset{\sim}{F}} Cl_T\,(F)\ .$$

1.41 Definition : $\underset{\sim}{F}$ is said to converge to a point p in T iff every nbhd. of p in T contains a member of $\underset{\sim}{F}$.

1.42 Definition : A Tychonoff space (X,τ) is realcompact iff any one of the following equivalent conditions holds.

(i) There exists no space Y which contains X as a dense, proper subspace and has the property that each continuous real-valued map f defined on X admits a continuous extension over Y.

(ii) (E. Hewitt [14]). Every real Z-ultrafilter is fixed (i.e. the intersection of all its members is non-void)

(iii) (Gillman and Jerison [12]). Every real prime Z-filter is fixed.

1.43 Definition : A topological space Y is a realcompactification of a Tychonoff space X iff Y is realcompact and contains a dense, homeomorphic copy of X.

1.44 Theorem (Hewitt [14]). Every Tychonoff space X has a realcompactification υX (called the Hewitt realcompactification of X) contained in βX, with the following equivalent properties.

(1) Every continuous map τ from X into any realcompact space Y admits a continuous extension τ^{υ} from υX into υY.

(2) For any sequence $\{Z_n\}$ of zero-sets in X

$$\mathrm{Cl}_{\upsilon X} \bigcap_{n=1}^{\infty} Z_n = \bigcap_{n=1}^{\infty} \mathrm{Cl}_{\upsilon X} Z_n .$$

(3) Every point of υX is the limit of a unique real Z-ultrafilter on X.

Furthermore, the space υX is unique, in the sense that if a realcompactification T of X satisfies any one, and hence all, of the above mentioned conditions, then there exists a homeomorphism of υX onto T which leaves X pointwise fixed.

1.45 Remark : υX is the smallest realcompact space between X and βX. In particular, X is realcompact iff $X = \upsilon X$.

1.46 Definition : (Alo and Shapiro [3]). If $\mathcal{L}$ is a c.p. normal base on X, we define the space $\eta(X,\mathcal{L})$ to be the set of all real $\mathcal{L}$-ultrafilters

on X with the W-topology. (where no confusion is possible, we shall denote $W(X,\mathcal{L})$ and $\eta(X,\mathcal{L})$ simply by $W(\mathcal{L})$ and $\eta(\mathcal{L})$ respectively).

1.47 Remark : Alo and Shapiro [3] have also proved that the family $\mathcal{L}^* = \{Cl_{\eta(\mathcal{L})}(L) : L \in \mathcal{L}\}$ corresponding to a normal countably productive base $\mathcal{L}$ on X, is a normal base in $\eta(\mathcal{L})$. Further, every real $\mathcal{L}^*$-ultrafilter on $\eta(\mathcal{L})$ is fixed, and in light of this fact, they have called the space $\eta(\mathcal{L})$ as $\mathcal{L}^*$-realcompact. It is also in line here to note that, in general, $\eta(\mathcal{L})$ is not realcompact. But if $\mathcal{L} = Z$, then $\eta(\mathcal{L})$ is not only realcompact but also $\eta(\mathcal{L}) = \upsilon X$.

We shall close this chapter by recalling some of the well-known results concerning a Hausdorff Wallman compactification $W(\mathcal{L})$ of a Tychonoff space X, $\mathcal{L}$ being a normal base on X. (see Alo and Shapiro [1], Banashewski [5]).

1.48 Proposition: If $\mathcal{L}$ is a normal base on X, then :

(a) $Cl_{W(\mathcal{L})}(L_1 \cap L_2) = Cl_{W(\mathcal{L})}(L_1) \cap Cl_{W(\mathcal{L})}(L_2)$, for all L_1, L_2 in $\mathcal{L}$.

(b) $\mathcal{L}^* = \{Cl_{W(\mathcal{L})}(L) : L \in \mathcal{L}\}$ is a base for closed sets in $W(\mathcal{L})$.

(c) $Cl_{W(\mathcal{L})}(L_1 \cup L_2) = Cl_{W(\mathcal{L})}(L_1) \cup Cl_{W(\mathcal{L})}(L_2)$, for all L_1, L_2 in $\mathcal{L}$.

(d) For each $p \in W(\mathcal{L})$, and for each nbhd. V of p in $W(\mathcal{L})$, there exists an $L \in \mathcal{L}$ such that $p \in Cl_{W(\mathcal{L})}(L) \subset V$. Hence, if $p, q \in W(\mathcal{L})$, $p \neq q$, then there exist L_p, L_q in $\mathcal{L}$ such that $p \in Cl_{W(\mathcal{L})}(L_p)$, $q \in Cl_{W(\mathcal{L})}(L_q)$ and $L_p \cap L_q = \phi$.

(e) $Cl_{W(\mathcal{L})}(L) \cap w(X) = w(L)$, for every L in $\mathcal{L}$, where w is the Wallman map of X into $W(\mathcal{L})$.

CHAPTER 2

LODATO SPACES

With the information of Chapter 1 in hand, we are now in a position to begin with our main results. In the present chapter, our primary concern will be to deal with the problem of extending a continuous function on a topological space, to a superspace in which it is dense. Our fundamental theorem in this direction is that a sufficient condition for a map $f : (X,\delta_1) \to (Y,\delta_2)$, (X,δ_1), (Y,δ_2) being LO-spaces, to have a continuous extension f_Σ from Σ_X to Σ_Y (where Σ_Z denotes the space of all bunches over (Z,δ) with a special topology, called the A-topology here) is that f is p-continuous. We will also obtain a necessary and sufficient condition for a separated LO-space to be compact and prove that every separated LO-space has a T_1-compactification. Finally, we will obtain for LO-spaces a satisfactory generalization of the well-known Smirnov Compactification Theorem known for EF-spaces.

The following result will have frequent applications in the subsequent work.

2.1 <u>Lemma</u> : Let $\underset{\sim}{F}$ be a ring of subsets of a non-void set X. Suppose $\underset{\sim}{P} \subset \underset{\sim}{F}$ such that :

(i) $\phi \notin \underset{\sim}{P}$

(ii) For A,B in $\underset{\sim}{F}$, $(A \cup B) \in \underset{\sim}{P}$ iff $A \in \underset{\sim}{P}$ or $B \in \underset{\sim}{P}$,

and (iii) $A \in \underset{\sim}{P}$, $A \subset B \in \underset{\sim}{F}$ implies that $B \in \underset{\sim}{P}$.

Then given an $A_o \in \underset{\sim}{P}$, there exists a prime $\underset{\sim}{F}$-filter $\underset{\sim}{L}$ on X such that $A_o \in \underset{\sim}{L} \subset \underset{\sim}{P}$. Further, if $\underset{\sim}{F} = P(X)$, then $\underset{\sim}{L}$ is an ultrafilter.

Proof. For $\underset{\sim}{P}$ satisfying (i), (ii) and (iii), define $\mathcal{A}$ to be the collection of all families $\underset{\sim}{A} \subset \underset{\sim}{P}$ such that :

(a) $A_o \in \underset{\sim}{A}$

(b) $A_i \in \underset{\sim}{A}$, $1 \leq i \leq n$ implies that $\bigcap_{i=1}^{n} A_i \in \underset{\sim}{P}$.

Partially order $\mathcal{A}$ by inclusion. Then clearly, every chain $\mathcal{C}$ in $\mathcal{A}$ has a maximal element, namely the union of all its members. Hence, by Zorn's Lemma, there exists a maximal element $\underset{\sim}{L}$ in $\mathcal{A}$. That $\underset{\sim}{L}$ is non-void and $\phi \notin \underset{\sim}{L}$ is obvious. If $A, B \in \underset{\sim}{L}$, then $A \cap B \in \underset{\sim}{P}$ and from the maximality of $\underset{\sim}{L}$ it follows that $(A \cap B) \in \underset{\sim}{L}$. Also, if $A \in \underset{\sim}{L}$, $A \subset D \in \underset{\sim}{F}$, then $D \in \underset{\sim}{P}$ and hence, once again as $\underset{\sim}{L}$ is maximal, it follows that $D \in \underset{\sim}{L}$. Thus $\underset{\sim}{L}$ is an $\underset{\sim}{F}$-filter on X. To see that $\underset{\sim}{L}$ is prime, suppose $A, B \in \underset{\sim}{F} - \underset{\sim}{L}$. Then there exist A_1, B_1 in $\underset{\sim}{L}$ such that $A \cap A_1$, $B \cap B_1$ are not in $\underset{\sim}{P}$. Setting $E = A_1 \cap B_1$, we note that $E \in \underset{\sim}{L}$, but, in view of (ii), $(A \cup B) \cap E \notin \underset{\sim}{P}$. Hence $(A \cup B) \notin \underset{\sim}{L}$, thereby showing that $\underset{\sim}{L}$ is a prime $\underset{\sim}{F}$-filter. Finally, when $\underset{\sim}{F} = P(X)$, then as every prime filter is also an ultrafilter, the result is obvious.

2.2 Proposition : In an EF-space (X,δ), a non-void family σ of subsets of X is a cluster iff it is a maximal bunch.

Proof : In view of Remark 1.24, we need only show that if (X,δ) is an EF-space, then every maximal bunch σ over (X,δ) is a cluster. We first observe the following :-

(i) Because of (B.1) and (P.3), $\phi \notin \sigma$.

(ii) $A \in \sigma$, $A \subset B \in P(X)$ implies that $B \in \sigma$ (1.26(1))

(iii) $(A \cup B) \in \sigma$ iff $A \in \sigma$ or $B \in \sigma$ (from (B.2)).

Hence, by Lemma 2.1, if $A_0 \in \sigma$, then there exists an ultrafilter $\underset{\sim}{L}$ on X such that $A_0 \in \underset{\sim}{L} \subset \sigma$. Applying (B.1), it immediately follows that $\sigma \subset \sigma(\underset{\sim}{L}) = \{A \in P(X) : A \,\delta\, L \text{ for every } L \text{ in } \underset{\sim}{L}\}$. But from Prop. 1.29, $\sigma(\underset{\sim}{L})$ is a cluster over (X,δ). Hence, by maximality of the bunch $\sigma, \sigma = \sigma(\underset{\sim}{L})$.

2.3 Proposition : In an EF-space (X,δ), every bunch is contained in a unique cluster.

Proof. If b is a bunch over (X,δ), then, by Prop. 1.25 and Prop. 2.2, b is contained in a cluster. To show the uniqueness, suppose on the contrary, that b is contained in two different clusters σ_1 and σ_2. Then, there exist $A_i \in P(X)$ such that $A_i \in \sigma_i$, $i = 1,2$ and $A_1 \not\delta A_2$. By (P.6), there is an $E \subset X$ such that $A_1 \not\delta E$, and $(X-E) \not\delta A_2$. Hence $E \notin \sigma_1$ and $(X-E) \notin \sigma_2$. Therefore $E \notin b$ and $(X-E) \notin b$, a contradiction to Prop. 1.26 (2).

2.4 Definition : Let σ be some distinguished family of subsets of a non-void set X, and let Σ be a collection of such σ. For $\underset{\sim}{A} \subset \Sigma$, we say that an $E \in P(X)$ absorbs $\underset{\sim}{A}$ iff $E \in \sigma$ for each σ in A. Also, for each $\underset{\sim}{A} \in P(\Sigma)$, we define $Cl(\underset{\sim}{A})$ as : $Cl(\underset{\sim}{A}) = \{\sigma \in \Sigma : E \text{ absorbs } \underset{\sim}{A} \text{ implies that } E \in \sigma\}$.

2.5 Lemma : (Lodato [19]) Let (X,δ) be a LO-space, and let Σ_X be the family of all bunches over (X,δ). If $\Sigma \subset \Sigma_X$, then the 'Cl' operator defined on $P(\Sigma)$ as in Def. 2.4 is a Kuratowski closure operator.

2.6 Definition : The topology τ_A induced on Σ by the above Kuratowski closure operator 'Cl' is called the Absorption topology (or the A-topology) on Σ.

2.7 Lemma : The A-topology τ_A on $\Sigma \subset \Sigma_X$ is T_1 iff $\sigma_1, \sigma_2 \in \Sigma$, $\sigma_1 \neq \sigma_2$ implies that $\sigma_1 \not\subset \sigma_2$ and $\sigma_2 \not\subset \sigma_1$.

Proof : The result is immediate from the fact that $\sigma_1 \in Cl(\sigma_2)$ iff $\sigma_1 \supset \sigma_2$.

Lodato, in fact, has proved the following:

2.8 Lemma : (Lodato [19]). The A-topology τ_A on $\Sigma \subset \Sigma_X$ is Hausdorff if either $A \in \sigma_1$ or $B \in \sigma_2$, σ_1, σ_2 in Σ, for $A,B \in P(X)$ such that $A \cup B = X$, implies that $\sigma_1 = \sigma_2$.

2.9 Theorem : Let (X,δ) be a LO-space and for $x \in X$, let $\Phi = \Phi_X : X \to \Sigma_X$ be a map defined by $\Phi(x) = \sigma_x$, the point cluster (see 1.26(3)). Then $\Phi : (X,\tau(\delta)) \to (\Sigma_X, \tau_A)$ is continuous and closed, and $\Phi(X)$ is dense in Σ_X. If δ is also separated, then X is homeomorphic to $\Phi(X)$.

Proof : That Φ is continuous and closed follows from the fact that $x \,\delta\, A$ iff $A \in \sigma_x$, i.e., $x \in A^-$ iff $\sigma_x \in Cl(\Phi(A))$. If δ is separated, then $x \neq y$ implies $\sigma_x \neq \sigma_y$ and hence Φ is one-to-one. Thus Φ is a homeomorphism between X and $\Phi(X)$. Finally, as $Cl(\Phi(X)) = \{\sigma \in \Sigma_X : X \in \sigma\} = \Sigma_X$ (from 1.26(1)) and hence $\Phi(X)$ is dense in Σ_X.

2.10 Corollary : If (i) $\Phi(X) \subset \Sigma \subset \Sigma_X$

(ii) $A \,\delta\, B$ implies there exists a $\sigma \in \Sigma$ such that $A,B \in \sigma$

and (iii) the A-topology τ_A on Σ is T_1,

then Φ is a proximal isomorphism betwen X and $\Phi(X)$, the latter having the subspace LO-proximity δ_X derived from (Σ,δ_o).

Proof : As (iii) ensures that Φ is one-to-one, (see Lemma 2.7) in order to prove that Φ is a proximal isomorphism, we have only to show that $A \,\delta\, B$ in X iff $Cl(\Phi(A)) \,\delta_o\, Cl(\Phi(B))$ in Σ . But this easily follows from the fact that $A \,\delta\, B$ iff there exists a $\sigma \in \Sigma$ such that $A,B \in \sigma$ iff there exists a $\sigma \in Cl(\Phi(A)) \cap Cl(\Phi(B))$ iff $Cl(\Phi(A)) \,\delta_o\, Cl(\Phi(B))$.

2.11 <u>Theorem:</u> (<u>The Fundamental Extension Theorem</u>)

Let (X,δ_1), (Y,δ_2) be LO-spaces and let $f : (X,\delta_1) \to (Y,\delta_2)$ be p-continuous. Then there exists an associated continuous map $f_\Sigma : (\Sigma_X,\tau_A) \to (\Sigma_Y,\tau_A)$, ($\tau_A$ being the A-topology) defined by $f_\Sigma(\sigma) = \{E \subset Y : f^{-1}(E^-) \varepsilon\, \sigma\}, \sigma \,\varepsilon\, \Sigma_X$. Further, $f_\Sigma(\sigma_x) = \sigma_{f(x)}$ and hence if δ_1 and δ_2 are also separated, then f_Σ may be considered as a continuous extension of f as follows :

$$\begin{tikzcd}
\Sigma_X \arrow[r, "f_\Sigma"'] & \Sigma_Y \\
X \arrow[u, "\bar{\Phi}_X"] \arrow[r, "f"'] & Y \arrow[u, "\bar{\Phi}_Y"']
\end{tikzcd}$$

<u>Proof</u> : We first show that if $\sigma \,\varepsilon\, \Sigma_X$, then $f_\Sigma(\sigma) \varepsilon\, \Sigma_Y$ by verifying (B.1), (B.2) and (B.3). (i) If $A,B \,\varepsilon\, f_\Sigma(\sigma)$, then $f^{-1}(A^-), f^{-1}(B^-) \varepsilon\, \sigma$ which implies that $f^{-1}(A^-)\ \delta_1\ f^{-1}(B^-)$ and since f is p-continuous, $A^-\ \delta_2\ B^-$. By 1.13, $A\ \delta_2\ B$. Hence (B.1) holds. (ii) $(A \cup B) \varepsilon\, f_\Sigma(\sigma)$ iff $f^{-1}[(A \cup B)^-] \varepsilon\, \sigma$ iff $f^{-1}(A^-) \cup f^{-1}(B^-) \varepsilon\, \sigma$ iff $f^{-1}(A^-) \varepsilon\, \sigma$ or $f^{-1}(B^-) \varepsilon\, \sigma$ iff $A \,\varepsilon\, f_\Sigma(\sigma)$ or $B \,\varepsilon\, f_\Sigma(\sigma)$, thus verifying (B.2). (iii) $A \,\varepsilon\, f_\Sigma(\sigma)$ iff $f^{-1}(A^-) \varepsilon\, \sigma$ iff $A^- \,\varepsilon\, f_\Sigma(\sigma)$, and hence (B.3) also holds. To show that f_Σ is continuous, we must show that if $\sigma \,\varepsilon\, Cl(\mathcal{A})$, $\mathcal{A} \,\varepsilon\, P(\Sigma)$, then $f_\Sigma(\sigma) \,\varepsilon\, Cl(f_\Sigma(\mathcal{A}))$. If not, then there exists a set $E \subset Y$ which absorbs $f_\Sigma(\mathcal{A})$, but does not belong to $f_\Sigma(\sigma)$. Hence $f^{-1}(\bar{E})$ absorbs $\mathcal{A}$ but it does not belong to σ, i.e., $\sigma \notin Cl(\mathcal{A})$, a contradiction. To show that $f_\Sigma(\sigma_x) = \sigma_{f(x)}$ for each $x \,\varepsilon\, X$, let $x \,\varepsilon\, X$. Then $f_\Sigma(\sigma_x) = \{A \,\varepsilon\, P(Y) : f^{-1}(A^-) \,\varepsilon\, \sigma_x\}$

$$= \{A \in P(Y) : x \; \delta_1 \; f^{-1}(A^-)\}$$

$$= \{A \in P(Y) : x \in f^{-1}(A^-)\}$$

$$= \{A \in P(X) : f(x) \in A^-\}$$

$$= \sigma_{f(x)} .$$

Finally, if δ_1 and δ_2 are also separated, then, from Th. 2.9, we note that Φ_X, Φ_Y are homeomorphisms, and hence, by identifying X, Y with $\Phi_X(X)$, $\Phi_Y(Y)$ respectively, it follows that f_Σ is a continuous extension of f.

2.12 Theorem : Let X be a dense, separated LO-subspace of a LO-space (T,δ_1). [δ_1 need not be separated]. Then, if τ_A is the A-topology on Σ_X, the map $\Psi = \Psi_T : (T,\tau(\delta)) \to (\Sigma_X,\tau_A)$ defined by, for each $x \in T$, $\Psi(x) = \sigma^x = \{E \in P(X) : x \; \delta_1 \; E\}$ is continuous. If $x \in X$, then $\Psi(x) = \sigma_x$, the point cluster, i.e. $\Psi/X = \Phi_X$. Further, if T is T_3, then Ψ is a homeomorphism of T into Σ_X.

Proof : We first verify that $\sigma^x \in \Sigma_X$. Since X is dense in T, $X \in \sigma^x$ and hence σ^x is non-void.

(i) $A,B \in \sigma^x$ implies that $x \; \delta_1 \; A$, $x \; \delta_1 \; B$, i.e. $x \in A^- \cap B^-$ which, by (P.4) implies that $A^- \; \delta_1 \; B^-$ and hence by Prop. 1.13, $A \; \delta_1 \; B$. Thus (B.1) is verified.

(ii) As $(A \cup B) \in \sigma^x$ iff $(A \cup B) \; \delta_1 \; x$ iff $A \; \delta_1 \; x$ or $B \; \delta_1 \; x$ (from (P.2)) iff $A \in \sigma^x$ or $B \in \sigma^x$, (B.2) holds.

(iii) (B.3) follows easily as $A \in \sigma^x$ iff $x \; \delta_1 \; A$ iff $x \in A^-$ iff $A^- \in \sigma^x$.

Hence $\sigma^x \in \Sigma_X$. For the continuity of Ψ, we must show that if $E \in P(T)$,

and $x \in E^-$, then $\Psi(x) \in Cl(\Psi(E))$. Suppose not. Then there exists an $A \in P(X)$ such that A absorbs $\Psi(E)$ but $A \notin \Psi(x) = \sigma^x$. This, in turn, implies that $E^- \subset A^-$ and $x \notin A^-$, a contradiction. Thus Ψ is continuous. That $\Psi/X = \Phi_X$ is obvious from the definition. Finally, if T is T_3, and $x_1, x_2 \in T$, $x_1 \neq x_2$, then there exist disjoint nbhds. N_1, N_2 in T of x_1, x_2 respectively. Clearly, $(X \cap N_1) \in \sigma^{x_1} - \sigma^{x_2}$ and $(X \cap N_2) \in \sigma^{x_2} - \sigma^{x_1}$, i.e. $\sigma^{x_1} \neq \sigma^{x_2}$ and hence Ψ is one-to-one. To prove that Ψ is a homeomorphism, it is enough to prove that Ψ is closed. Let $F \in P(T)$, and $x \notin F^-$. Since T is T_3, there are disjoint nbhds. N_F, N_x in T of F, x respectively. Since X is dense in T, $(N_F \cap X)$ absorbs $\Psi(F)$ but does not belong to σ^x. Hence $\Psi(x) = \sigma^x \notin Cl(\Psi(F))$, i.e. Ψ is closed.

Leader [17] has shown that the Smirnov Compactification (see Smirnov [31]) $\underset{\sim}{X}$ of a separated EF-space (X,δ) is the family of all clusters in X with the A-topology, τ_A. From Prop. 2.3, we also know that to every bunch σ in Σ_X there corresponds a unique cluster σ_θ in X containing σ. Using these facts, we deduce the following result.

2.13 <u>Theorem</u> : If (X,δ) is a separated EF-space, then the map $\theta = \theta_X : (\Sigma_X, \tau_A) \to \underset{\sim}{X}$, given by $\theta(\sigma) = \sigma_\theta$ is continuous. Moreover, $\theta(\sigma_x) = \sigma_x$.

<u>Proof</u> : For $\sigma \in \Sigma_X$ and $\underset{\sim}{A} \subset P(\Sigma_X)$, if $\theta(\sigma) = \sigma_\theta \notin Cl(\theta(\underset{\sim}{A}))$ then, $\sigma_\theta \not\delta^* \theta(\underset{\sim}{A})$ [δ^* being the EF-proximity of $\underset{\sim}{X}$]. Hence, since $\underset{\sim}{X}$ is an EF-space, from Remark 1.4, there exist nbhds. N_1, N_2 of σ_θ, $\theta(\underset{\sim}{A})$ respectively, such that $N_1 \not\delta^* N_2$. Then clearly, $N_2 \cap X$ absorbs $\underset{\sim}{A}$, but is not in σ, i.e. $\sigma \notin Cl(\underset{\sim}{A})$. Hence θ is continuous. That $\theta(\sigma_x) = \sigma_x$ is immediate from the definition of θ.

2.14 Remark : Leader [17] has also proved that an EF-space (X,δ) is compact iff every cluster over (X,δ) is a point cluster (and hence converges to a unique point of X). Making use of this result, we shall derive the following corollary to Th. 2.13.

2.15 Corollary : If X is compact Hausdorff with the EF-proximity δ_o, then the map $\theta = \theta_X : (\Sigma_X, \tau_A) \to (X, \tau(\delta_o))$ given by $\theta(\sigma) = x_\sigma$, the unique point to which σ_θ converges, is continuous.

Proof : If X is compact Hausdorff, then $X = \underset{\sim}{X}$ (by identifying x with σ_x for each x in X) and hence, by Th. 2.13, θ is continuous.

2.16 Theorem : If (X,δ_o) is a T_3 LO-space and if $\Sigma \subset \Sigma_X$ such that each $\sigma \in \Sigma$ converges to a unique x_σ in X, then the map $\theta = \theta_X : (\Sigma, \tau_A) \to (X, \tau(\delta_o))$, given by $\theta(\sigma) = x_\sigma$ is continuous.

Proof : To prove that θ is continuous, we again show that for $\sigma \in Cl(\underset{\sim}{A})$, $\underset{\sim}{A} \in P(\Sigma)$, $\theta(\sigma) \in Cl(\theta(\underset{\sim}{A}))$. Suppose $\theta(\sigma) \notin Cl(\theta(\underset{\sim}{A}))$, then as X is T_3, there exist disjoint closed nbhds. N_1, N_2 in X of $\theta(\sigma) = x_\sigma$ and $\theta(\underset{\sim}{A})$ respectively. Clearly, N_2 absorbs $\underset{\sim}{A}$, but is not in σ. Hence $\sigma \notin Cl(\underset{\sim}{A})$, i.e. θ is continuous.

We next obtain a result for LO-spaces that corresponds to the result mentioned in Remark 2.14 for EF-spaces.

2.17 Theorem : A separated LO-space (X,δ) is compact iff every bunch $b(\underset{\sim}{L})$ generated by a closed ultrafilter $\underset{\sim}{L}$ on X (see Prop. 1.28) is a point cluster.

Proof : We note that $b(\underset{\sim}{L})$ is a point cluster σ_{x_o} for some x_o in X iff $\{x_o\} \in b(\underset{\sim}{L})$ iff $\{x_o\} \in \underset{\sim}{L}$ (because $\{x_o\}$ is closed in X) iff $\underset{\sim}{L}$ converges to x_o. Hence every $b(\underset{\sim}{L})$ over (X,δ) is a point cluster iff every closed ultrafilter $\underset{\sim}{L}$ is convergent iff X is compact.

We will now show that every separated LO-space has a T_1-compactification. In view of Remark 2.14 and Prop. 2.2 , this result is obviously a partial generalization of the Smirnov Theorem.

2.18 Theorem : Let (X,δ) be a separated LO-space and let X^* be the family of all maximal bunches in X with the A-topology τ_A. Then X^* is a compact T_1-space containing a dense homeomorphic copy of X.

Proof : As X^* is the family of all maximal bunches over (X,δ), from Lemma 2.7, it follows that X^* is a T_1-space. Using Th. 2.9, we see that X is homeomorphic to $\Phi(X)$, which is dense in X^*. So we need only prove that X^* is also compact, for which it is enough to prove that if $\{A^*_\alpha : \alpha \in \mathcal{A}\}$ [where $A^*_\alpha = \{\sigma \in X^* : A_\alpha \in \sigma,\ A_\alpha$ closed in $X\}$] has the finite intersection property (f.i.p.), then $\bigcap_{\alpha\in\mathcal{A}} A^*_\alpha \neq \phi$. [This is due to the fact that the family $\{A^*_\alpha : A_\alpha$ closed in $X\}$ is a base for closed sets in (Σ_X,τ_A)]. Since $\{A^*_\alpha : \alpha \in \mathcal{A}\}$ has the f.i.p., the corresponding family $\underset{\sim}{F} = \{A_\alpha : \alpha \in \mathcal{A}\}$ of closed subsets of X has the property that every finite subfamily of $\underset{\sim}{F}$ is a subclass of some $\sigma \in X^*$.

Let $\mathcal{G}$ be the family of all collections $\underset{\sim}{G}$ of closed subsets of X such that

(i) $\underset{\sim}{F} \subset \underset{\sim}{G}$

and (ii) $G_i \in \underset{\sim}{G}$, $1 \leq i \leq n$, implies there exists a $\sigma \in X^*$ such that $G_i \in \sigma$, $1 \leq i \leq n$.

We now order the family $\mathcal{G}$ partially by inclusion. If $\mathcal{C}$ is a chain in $\mathcal{G}$, then it is clear that $\mathcal{C}$ has an upper bound, namely the union of all its members. Hence, applying Zorn's Lemma, $\mathcal{G}$ has a maximal element $\underset{\sim}{M}$.

Let $b(\underset{\sim}{M}) = \{E \in P(X) : E^- \in \underset{\sim}{M}\}$. Then $b(\underset{\sim}{M})$ is a bunch over (X,δ), as the following argument shows :

(a) $A, B \in b(\underset{\sim}{M})$ implies $A^-, B^- \in \underset{\sim}{M}$, and since $\underset{\sim}{M}$ satisfies (ii), $A^-, B^- \in \sigma$ for some σ in X^* and hence $A^- \,\delta\, B^-$. Using Prop. 1.13, $A \,\delta\, B$., verifying (B.1).

(b) $A \notin b(\underset{\sim}{M})$, $B \notin b(\underset{\sim}{M})$ implies there exist A_i, $1 \leq i \leq n$ and $B_j, 1 \leq j \leq m$, in $b(\underset{\sim}{M})$ such that no σ in X^* contains either $\{A\} \cup \{A_i\}_{i=1}^{n}$ or $\{B\} \cup \{B_j\}_{j=1}^{m}$. Because of (B.2) this in turn implies that there is no σ in X^* such that $A \cup B$, A_i, B_j, $i = 1,\ldots,n$, $j = 1,\ldots,m$ all belong to it. Hence $(A \cup B) \notin b(\underset{\sim}{M})$. Conversely, $(A \cup B) \notin b(\underset{\sim}{M})$ implies that there exist E_k, $1 \leq k \leq \ell$ in $b(\underset{\sim}{M})$ such that $\{(A \cup B)\} \cup \{E_k\}_{k=1}^{\ell} \not\subset \sigma$ for all σ in X^*. Hence, by (B.2), $\{A\} \cup \{E_k\}_{k=1}^{\ell} \not\subset \sigma$ and $\{B\} \cup \{E_k\}_{k=1}^{\ell} \not\subset \sigma$ for all σ in X^*, i.e. $A \notin b(\underset{\sim}{M})$ and $B \notin b(\underset{\sim}{M})$. Thus (B.2) holds in $b(\underset{\sim}{M})$.

(c) $A \in b(\underset{\sim}{M})$ iff $A^- \in \underset{\sim}{M}$ iff $A^- \in b(\underset{\sim}{M})$ as $\underset{\sim}{M} \subset b(\underset{\sim}{M})$.
Hence (B.3) is also valid for $b(\underset{\sim}{M})$, i.e. $b(\underset{\sim}{M}) \in \Sigma_X$. By Prop. 1.25, $b(\underset{\sim}{M}) \subset \sigma_0$ for some σ_0 in X^*. Clearly, $\sigma_0 \in \bigcap_{\alpha \in \mathcal{A}} A^*_\alpha$ and hence X^* is compact.

We now apply the above results to obtain the theorem of Lodato [19].

2.19 Lodato's Theorem : Given a set X and a binary relation δ on $P(X)$, the following are equivalent :-

(a) There exists a T_2-space Y in which X is embedded such that

$A \,\delta\, B$ in X iff $A^- \cap B^- \neq \emptyset$ in Y.

(b) (X,δ) is a separated LO-space possessing a family $\underset{\sim}{B}$ of bunches such that :

(i) $A \,\delta\, B$ implies there is a σ in $\underset{\sim}{B}$ containing A and B

and (ii) $\sigma_1, \sigma_2 \in \underset{\sim}{B}$ and either $A \in \sigma_1$ or $B \in \sigma_2$ for all A, B in $P(X)$ such that $A \cup B = X$, then $\sigma_1 = \sigma_2$.

Proof : The result follows from Lemma 2.8, Corollary 2.10 and the fact that $A \delta B$ iff $Cl(\bar{\Phi}(A)) \cap Cl(\bar{\Phi}(B)) \neq \phi$ (as proved earlier in 2.10).

We now recall Smirnov's Theorem (see Smirnov [31]).

2.20 Smirnov's Theorem : Let (X,δ) be a separated EF-space. Then

(i) there exists a compact T_2-space $\underset{\sim}{X}$ containing a dense homeomorphic copy of X.

(ii) $A \delta B$ iff $\bar{A} \cap \bar{B} \neq \phi$ in $\underset{\sim}{X}$,

(iii) If (Y,δ') is another separated EF-space, and $f : (X,\delta) \to (Y,\delta')$ is p-continuous, then f has a continuous extension $\underset{\sim}{f} : \underset{\sim}{X} \to \underset{\sim}{Y}$.

Further,

(iv) any space $\underset{\sim}{X}$ satisfying (i) and (ii) is unique upto proximal isomorphism, and can be described as the space of all clusters over (X,δ) with the A-topology τ_A.

Our generalization of the above theorem for LO-spaces reads as follows :

2.21 Theorem : Let (X,δ) be a separated LO-space such that if $A \delta B$, then there exists a bunch over (X,δ) which contains A and B. Then :

(i) there exists a compact T_1-space $\underset{\sim}{X}$ (the space of all maximal bunches over (X,δ) with the A-topology τ_A), containing a dense, homeomorphic copy of X.

(ii) $A \delta B$ iff $Cl(\bar{\Phi}(A)) \cap Cl(\bar{\Phi}(B)) \neq \phi$ in $\underset{\sim}{X}$.

(iii) if (Y,δ') is another separated LO-space and if $f : (X,\delta) \to (Y,\delta')$ is p-continuous, then f has a continuous extension $f_\Sigma : \underset{\sim}{X} \to (\Sigma_Y,\tau_A)$.

Proof : If we set $\underset{\sim}{X}$ to be the family of all maximal bunches over (X,δ), with the A-topology τ_A, then, by Th. 2.18, $\underset{\sim}{X}$ is a compact T_1-space

containing a dense homeomorphic copy of X. That $A \delta B$ iff $Cl(\bar{\Phi}(A)) \cap Cl(\bar{\Phi}(B)) \neq \phi$ in $\underset{\sim}{X}$ follows from Corollary 2.10, by noting that each bunch is contained in a maximal bunch (Prop. 1.25). Finally, (iii) follows from the Fundamental Extension Theorem (2.11). [note that $\underset{\sim}{X} \subset \Sigma_X$].

2.22 Remark : No doubt we have lost the uniqueness of $\underset{\sim}{X}$ in the above theorem, but this we cannot hope for without requiring $\underset{\sim}{X}$ to be Hausdorff, in which case, $\underset{\sim}{X}$ becomes an EF-space .

2.23 Remark : We now explain how Th. 2.21 generalizes Smirnov's Theorem. In every EF-space, $A \delta B$ implies that there exists a cluster σ which contains both A and B. (see Leader [17]). Also, in that case, Prop. 2.2 shows that $\underset{\sim}{X}$ is the Smirnov compactification of X. Finally, suppose δ , δ' are separated proximities and $f : (X,\delta) \to (Y,\delta')$ is p-continuous. Then, by Th. 2.13, the map $\theta_Y : (\Sigma_Y, \tau_A) \to \underset{\sim}{Y}$ which assigns to each bunch in Σ_Y the unique cluster containing it is continuous. Hence f has an extension $\underset{\sim}{f} : \underset{\sim}{X} \to \underset{\sim}{Y}$ given by $\underset{\sim}{f} = \theta_Y \circ f_\Sigma$.

CHAPTER 3.

CONTINUOUS EXTENSIONS OF MAPS FROM DENSE SUBSPACES.

Our results in the previous chapter were partly motivated by some well-known extension theorems. However, during the course of the present investigation, a general technique (based on the results already discussed in chapter 2) of solving extension problems was discovered. In this chapter, we shall fruitfully employ the same to obtain alternate proofs of theorems such as those of Taimanov [37] and McDowell [21]. At the same time, new extension theorems will also be developed. Thus, if X is a dense subspace of a topological space T, then we shall obtain necessary and/or sufficient conditions for a continuous map $f : X \to Y$ to have a continuous extension $\overline{f} : T \to Y$, when the range space is T_3 and satisfies some axiom which is 'weaker' than that of compactness. For example, if Y is locally compact Hausdorff, and if X is a T_1-dense subspace of an R_o-space T, then we will prove that f has a continuous extension iff (i) for every pair of disjoint closed sets F_1, F_2 in Y, at least one of which is compact, $Cl_T\, f^{-1}(F_1) \cap Cl_T\, f^{-1}(F_2) = \phi$ and (ii) for each $t \in T$, there exists a compact subset C_t of Y such that $t \in Cl_T\, f^{-1}(C_t)$. Similarly, we shall consider the cases when Y is paracompact, metacompact, countably compact, countably paracompact and countably metacompact. Finally, we will generalize Blefko's Theorem D [6] to the case when Y is T_3.

We begin the section with the following Lemma.

3.1 <u>Lemma</u> : Let X be a T_1-dense subspace of a LO-space (T,δ_0) and let Y be dense in a separated LO-space $(\underset{\sim}{Y},\delta_0)$. If a map $f : X \to Y$ has a continuous extension $\bar{f} : (T,\tau(\delta_0)) \to (\underset{\sim}{Y},\tau(\delta_0))$, then f is p-continuous w.r.t. the subspace LO-proximities induced on X,Y by (T,δ_0), $(\underset{\sim}{Y},\delta_0)$ respectively.

<u>Proof</u> : If f has a continuous extension $\bar{f} : (T,\tau(\delta_0)) \to (\underset{\sim}{Y},\tau(\delta_0))$, then, from Prop. 1.20, $\bar{f}$ is p-continuous and so is its restriction $f = \bar{f}/_X$.

Taimanov's result [37] reads as follows :

3.2 <u>Taimanov's Theorem</u> : If X is dense in a T_1-space T, and Y is compact Hausdorff, then a continuous map $f : X \to Y$ has a continuous extension $\bar{f} : T \to Y$ iff for disjoint closed sets F_1,F_2 in Y, $Cl_T\, f^{-1}(F_1) \cap Cl_T\, f^{-1}(F_2) = \phi$.

3.3 <u>Remark</u> : We can interpret the above theorem proximally as follows : If T and Y are assigned the LO-proximity δ_0 and the EF-proximity δ_0 respectively, then f has a continuous extension iff f is p-continuous.

As we shall soon prove, Taimanov's Theorem can be obtained as a corollary to the following theorem, which is more general.

3.4 <u>Theorem</u> : Let X be a T_1-dense subspace of an R_0-space T and let X be assigned the subspace proximity induced by the LO-proximity δ_0 on T. Let (Y,δ) be a separated EF-space and $\underset{\sim}{Y}$ its Smirnov Compactification. Then a continuous function $f : X \to Y$ has a continuous extension $\bar{f} : T \to \underset{\sim}{Y}$ iff f is p-continuous.

<u>Proof</u> : Necessity follows from Lemma 3.1. To prove the sufficiency, let f be p-continuous. Consider the following diagram.

$$\begin{tikzcd}
 & \Sigma_X \arrow[rr, "f_\Sigma"] \arrow[drr, phantom, "2.11"] & & \Sigma_Y \arrow[dr, "\theta_Y", "2.13"'] & \\
T \arrow[ur, "\Psi_T", "2.12"'] & & & & \underset{\sim}{Y} \\
 & X \arrow[ul, "1_X"] \arrow[uu, "\Phi_X"', "2.9"] \arrow[rr, "f"'] & & Y \arrow[uu, "\Phi_Y"', "2.9"] &
\end{tikzcd}$$

Then, clearly, $\bar{f} : \theta_Y \circ f_\Sigma \circ \Psi_T : T \to \underset{\sim}{Y}$ is a continuous extension of f.

3.5 <u>Remark</u> : If Y is compact Hausdorff, then Y is homeomorphic to $\underset{\sim}{Y}$ and we may consider $\bar{f}$ to be a map from T into Y. This proves the sufficiency condition of Taimanov's Theorem (see 3.2 and 3.3). Necessity follows from Lemma 3.1 as before.

McDowell [21] has proved the following extension theorem.

3.6 <u>McDowell's Theorem</u> : Given Tychonoff spaces X, Y, $X \subset E$, X dense in E and such that functionally separated sets in X have functionally separated closures in E, then every continuous map $f : X \to Y$ may be continuously extended to a mapping $f' : E_Y \to Y$ defined by, for $e \in E$, $f'(e) = y$ iff $e \in E_y$. Moreover, E_Y is the largest subspace of E to which f has a continuous extension.

3.7 <u>Remark</u> : In the above theorem, for $y \in Y$, we define

$$E_y = \bigcap \{Cl_E\, f^{-1}(V) : V \text{ a nbhd. of } y \text{ in } Y\} \quad \text{and } E_Y = \bigcup_{y \in Y} E_y .$$

In order to prove theorem 3.6 by the proximity approach, we will again prove a more general result.

3.8 <u>Theorem</u> : Let X be a T_1-dense subspace of an R_0-space E, and let functionally separated sets in X have disjoint closures in E. Let Y be a Tychonoff space. Then every continuous map $f : X \to Y$ can

be extended to a continuous mapping $\overline{f} : E_Y \to Y$ defined by, for $x \in E$, $\overline{f}(x) = y$ iff $x \in E_y$. Moreover, E_Y is the largest subspace of E to which f has a continuous extension.

Proof : We first show that under the present hypothesis, if E is assigned the LO-proximity δ_o and Y has any compatible EF-proximity, then every continuous map $f : X \to Y$ is also p-continuous. For let $C \not\delta D$ in Y. Then, by Th. 1.16, as $\delta_F > \delta$, C and D are functionally separated in Y. Since f is continuous, $f^{-1}(C)$ and $f^{-1}(D)$ are also functionally separated in X and hence $f^{-1}(C) \not\delta_o f^{-1}(D)$ in E, thereby implying that f is p-continuous. By Th. 3.4, f has a continuous extension $\overline{f} : E \to \underset{\sim}{Y}$. We now find $\overline{f}^{-1}(Y)$. Clearly, $\overline{f}(x) = y \in Y$ iff $f_\Sigma(\sigma^x)$ converges to y iff the nbhd. filter $\mathcal{N}_y \subset f_\Sigma(\sigma^x)$ iff $x \in \bigcap_{N_y \in \mathcal{N}_y} Cl_E f^{-1}(N_y)$ $= E_y$. Thus, $\overline{f}^{-1}(Y) = \bigcup_{y \in Y} E_y = E_Y$, and hence the result is proved.

3.9 Lemma : Let X be a T_1-dense subspace of an R_o-space T and let Y be a T_3-space. Assign to Y and X the δ_o LO-proximity and the derived subspace LO-proximity δ_X from (T,δ_o) respectively. Then a p-continuous map $f : X \to Y$ has a continuous extension $\overline{f} : T \to Y$ iff for each $t \in T$, the family $f_\Sigma(\sigma^t) = \{A \in P(Y) : t \in Cl_T f^{-1}(A^-)\}$ converges to some unique point y^t in Y.

Proof : Sufficiency is the direct consequence of Th. 2.12, Th. 2.11 and Th. 2.16. To prove the necessity, let f have a continuous extension $\overline{f} : T \to Y$. For t in T, let $\overline{f}(t)=y^t$. Then denseness of X in T implies the existence of a net $\langle x_d : d \in D \rangle$ in X which converges to t. From the continuity of $\overline{f}$ it follows that the net $\langle \overline{f}(x_d) : d \in D \rangle = \langle f(x_d) : d \in D \rangle$ converges to $\overline{f}(t) = y^t$ in Y. Hence, if N_1 and N_2 are nbhds. of

t, y^t in T, Y respectively, then the net $\langle x_d : d \in D\rangle$ is eventually in $N_1 \cap f^{-1}(N_2)$ i.e. $t \in Cl_T\, f^{-1}(N_2)$ and so $N_2 \in f_\Sigma(\sigma^t)$. Thus $\mathcal{N}_y \subset f_\Sigma(\sigma^t)$, which implies that $f_\Sigma(\sigma^t)$ converges to y^t (Def. 1.30). Finally, to check the uniqueness of y^t, suppose there exist y_1, y_2 in Y, $y_1 \neq y_2$, such that $\mathcal{N}_{y_i} \subset f_\Sigma(\sigma^t)$, $i = 1,2$. Then, if G_1, G_2 are arbitrary nbhds. in Y of y_1, y_2 respectively, we have that $t \in Cl_T\, f^{-1}(G_1) \cap Cl_T\, f^{-1}(G_2)$ i.e. $Cl_T\, f^{-1}(G_1)\,\delta_o\, Cl_T\, f^{-1}(G_2)$ and hence, by the p-continuity of f it follows that $G_1\, \delta_o\, G_2$ in Y i.e. $\overline{G}_1 \cap \overline{G}_2 \neq \phi$. But this is impossible since Y is T_3 and hence every pair of distinct points has disjoint closed nbhds. Therefore $y_1 = y_2$.

The following Lemma is a slight variation of the above result.

3.10 <u>Lemma</u> : Let X be a separated dense subspace of a LO-space (T,δ_o) and let (Y,δ') be a separated EF-space. If a continuous map $f : X \to Y$ has a continuous extension $\overline{f} : T \to Y$, then :

(i) f is p-continuous.

(ii) for each $t \in T$, the family $f_\Sigma(\sigma^t) = \{A \in P(Y) : t \in Cl_T\, f^{-1}(\overline{A})\}$ converges to some unique y^t in Y.

<u>Proof</u> : (i) follows from Lemma 3.1, while the proof for (ii) is similar to the 'necessary' part in the proof of Lemma 3.9.

3.11 <u>Theorem</u> : Let X be a T_1-dense subspace of an R_o-space T and let Y be a locally compact Hausdorff space. Then a continuous map $f : X \to Y$ has a continuous extension $\overline{f} : T \to Y$ iff

(i) for every pair of disjoint closed set F_1, F_2 in Y, at least one of which is compact, $Cl_T f^{-1}(F_1) \cap Cl_T f^{-1}(F_2) = \phi$ and

(ii) for each $t \in T$, there exists a compact subset C_t of Y such that $t \in Cl_T\, f^{-1}(C_t)$.

Proof : We first note that in a locally compact space Y, the EF-proximity δ_Y induced by its Alexandroff one-point compactification $Y \cup \{\infty\}$ can be described as : $A \not\delta_Y B$ iff $\overline{A} \cap \overline{B} = \phi$ and at least one of $\overline{A}$, $\overline{B}$ is compact (the closures being taken in Y). Let X have the induced subspace proximity δ_X from the LO-space (T, δ_o) and assign Y the subspace δ_Y proximity as given above. Then, as every point y in Y has a compact nbhd., the necessity of (i) and (ii) is a direct consequence of Lemma 3.10. To prove the sufficiency, as condition (i) implies that f is p-continuous, from Th. 3.4, f has a continuous extension $\overline{f} : T \to Y \cup \{\infty\}$ (Note that $Y \cup \{\infty\}$ is the Smirnov compactification of Y). To show that $\overline{f}(T) \subset Y$, it is enough to prove that for no $t \in T$, $f_\Sigma(\sigma^t)$ converges to the point ∞. Let $t \in T$. Then, from (ii), there exists a compact C_t in Y such that $C_t \in f_\Sigma(\sigma^t)$. But as $Y \cup \{\infty\}$ is the Alexandroff one-point compactification of Y, there exists an open set G in $Y \cup \{\infty\}$ such that $\infty \in G \subset \overline{G} \subset (Y - C_t)$, which implies that $G \not\delta_Y C_t$ and as $C_t \in f_\Sigma(\sigma^t)$, from (B.1) it follows that $G \notin f_\Sigma(\sigma^t)$. Hence $f_\Sigma(\sigma^t)$ does not converge to ∞.

3.12 Theorem : Let X be a T_1-dense subspace of an R_o-space T and let Y be a paracompact Hausdorff (and hence T_3-) space. If T is such that

(E) : Every locally finite (or l.f.) open cover $\underset{\sim}{G}$ of X is closure-preserving in T,

then a continuous map $f : X \to Y$ has a continuous extension $\overline{f} : T \to Y$ iff for disjoint closed sets F_1, F_2 in Y, $Cl_T f^{-1}(F_1) \cap Cl_T f^{-1}(F_2) = \phi$.

Proof : Let T, Y be assigned the LO-proximity δ_o and let X have the induced subspace proximity δ_X from (T, δ_o). Then the above necessary

and sufficient condition reduces to the condition that f be p-continuous. Hence necessity follows from Lemma 3.1. To prove the sufficiency, in view of Lemma 3.9, it is sufficient to show that for each $t \in T$, the family $f_{\Sigma}(\sigma^t) = \{A \in P(Y) : t \in Cl_T\, f^{-1}(\bar{A})\}$ converges to some unique y^t in Y. Suppose not. Then, for each y in Y, there exists an open nbhd. N_y of y such that $N_y \notin f_{\Sigma}(\sigma^t)$, and hence $\underset{\sim}{M} = \{N_y : y \in Y,\ N_y$ an open nbhd. of y, $N_y \notin f_{\Sigma}(\sigma^t)\}$ is an open cover of Y. From the paracompactness of Y, there exists a l.f. open refinement $\{0_\alpha : \alpha \in A\}$ of $\underset{\sim}{M}$ which is a cover of Y. From the continuity of f, $\underset{\sim}{G} = \{f^{-1}(0_\alpha) : \alpha \in A\}$ is also an open cover of X, and as $\{0_\alpha : \alpha \in A\}$ is l.f., so is $\underset{\sim}{G}$. Hence by condition (E), $\underset{\sim}{G}$ is closure-preserving in T. Thus $\bigcup_{\alpha\in A} Cl_T f^{-1}(0_\alpha) = Cl_T(\bigcup_{\alpha\in A} f^{-1}(0_\alpha)) = Cl_T X = T$, which implies that there exists at least one α' in A such that $t \in Cl_T\, f^{-1}(0_{\alpha'})$ i.e. such that $0_{\alpha'} \in f_{\Sigma}(\sigma^t)$. But as $\{0_\alpha : \alpha \in A\}$ is a refinement of M, and no member of M is in $f_{\Sigma}(\sigma^t)$, from Prop. 1.26(1), no 0_α, $\alpha \in A$ can be in $f_{\Sigma}(\sigma^t)$, a contradiction. Hence, for each $t \in T$, $f_{\Sigma}(\sigma^t)$ converges to some y^t in Y. For the uniqueness of y^t, we note that if $y_1, y_2 \in Y$, $y_1 \neq y_2$, then there exist disjoint closed nbhds. N_1, N_2 of y_1, y_2 respectively, and hence $N_1 \not\delta N_2$. But $f_{\Sigma}(\sigma^t)$ is a bunch over (Y,δ_o) (Th. 2.11), and consequently, in view of the axiom (B.1), at least one N_i, $i = 1,2$ is not in $f_{\Sigma}(\sigma^t)$, and hence for each $t \in T$, y^t is unique.

3.13 <u>Theorem</u> : Let X be a T_1-dense subspace of an R_o-space T, and let Y be a metacompact T_3-space. If T is such that :

(E') : Every open point finite (p.f.) cover $\underset{\sim}{G}$ of X is closure-preserving in T,

then a continuous map $f : X \to Y$ has a continuous extension $\overline{f} : T \to Y$ iff for disjoint closed sets F_1, F_2 in Y, $Cl_T f^{-1}(F_1) \cap Cl_T f^{-1}(F_2) = \phi$.

Proof : The result is immediate from the proof of Th. 3.12, when the terms 'l.f.' and 'paracompact' are replaced by the terms 'p.f.' and 'metacompact' respectively, [note that the pre-images of p.f. families are also p.f.].

3.14 Remark : We now extend similar results to cases when the range space Y is countably compact, countably paracompact and countably metacompact respectively. In the proof of each of these, we shall be assigning the same proximities to the spaces as in Th. 3.12. Hence, the 'necessary' part of the proof will follow from Lemma 3.1., while for the 'sufficiency' as in Th. 3.12, it will be enough to prove that for each $t \in T$, the family $f_\Sigma(\sigma^t)$ converges to some unique y^t in Y.

3.15 Theorem : Let X be a T_1-dense subspace of an R_0-space T and let Y be a countably compact T_3-space. Let $f : X \to Y$ be a continuous map such that, for each t in T, the family $f_\Sigma(\sigma^t) = \{A \in P(Y) : t \in Cl_T f^{-1}(A^-)\}$ contains a closed countable subset C_t of Y. Then f has a continuous extension $\overline{f} : T \to Y$ iff for disjoint closed sets F_1, F_2 in Y, $Cl_T f^{-1}(F_1) \cap Cl_T f^{-1}(F_2) = \phi$.

Proof : Let $t \in T$, and consider the family $\underset{\sim}{F} = \{F \in f_\Sigma(\sigma^t) :$ F closed in Y $\}$. Then clearly, $C_t \in \underset{\sim}{F}$. Also (i) $\phi \notin \underset{\sim}{F}$, (ii) A, B closed in Y and $(A \cup B) \in \underset{\sim}{F}$ implies, from (B.2), that either $A \in \underset{\sim}{F}$ or $B \in \underset{\sim}{F}$ and (iii) $A \in \underset{\sim}{F}$, $A \subset B$, B closed in Y, implies that $t \in Cl_T f^{-1}(A^-) \subset Cl_T f^{-1}(B^-)$ i.e. $B \in \underset{\sim}{F}$. Hence, from Lemma 2.1, there exists a prime closed filter $\underset{\sim}{P}$ on Y such that $C_t \in \underset{\sim}{P} \subset \underset{\sim}{F}$. We now prove that there

exists at least one point y^t in C_t such that $f_\Sigma(\sigma^t)$ converges to y^t. Suppose not. Then, for every $y_n \in C_t$, $n \in \mathbb{N}$, there exists an open nbhd. N_{y_n} of y_n such that $N_{y_n} \notin \underset{\sim}{P}$ and consequently, $\overline{N_{y_n}} \notin \underset{\sim}{P}$ either (from (B.3)). Also, as $(X-N_{y_n}) \cup \overline{N_{y_n}} = X \in \underset{\sim}{P}$, and $\underset{\sim}{P}$ is a prime closed filter in Y, it follows that, as $\overline{N_{y_n}} \notin \underset{\sim}{P}$, $(X - N_{y_n}) \in \underset{\sim}{P}$ for each y_n in C_t.

Clearly $\bigcap\{X-N_{y_n} : y_n \in C_t\} \cap C_t = \emptyset$. But the countable family $\{X-N_{y_n} : y_n \in C_t\} \cup \{C_t\}$ of closed sets in Y, being a subclass of $\underset{\sim}{P}$, has the f.i.p. and consequently, by the countable compactness of Y, must have a non-void total intersection (see Engelking [10]) a contradiction. Finally, the uniqueness of y^t follows as in Th. 3.12.

3.16 <u>Theorem</u> : Let X be a T_1-dense subspace of an R_o-space T and let Y be a countably paracompact T_3-space. If $f : X \to Y$ is a continuous map, and if :

(E_1) : every l.f. open cover $\underset{\sim}{G}$ of X is closure-preserving in T,

and (E_2) : for every $t \in T$, there exists a countable closed subset C_t of Y such that $t \in Cl_T\, f^{-1}(C_t)$, and $t \notin Cl_T\, f^{-1}(\overline{Y - C_t})$,

then f has a continuous extension $\overline{f} : T \to Y$ iff for disjoint closed sets F_1, F_2 in Y, $Cl_T\, f^{-1}(F_1) \cap Cl_T\, f^{-1}(F_2) = \emptyset$.

<u>Proof</u> : We first recall that Y is countably paracompact iff every countable open cover of Y has a l.f. open refinement which also covers Y (see Engelking [10]). Let $t \in T$, and suppose that $f_\Sigma(\sigma^t)$ does not converge to any y^t in Y, and hence, in particular, to any y^t in C_t. Then, for each $y_n \in C_t$, there exists an open nbhd. N_{y_n} of y_n such that $N_{y_n} \notin f_\Sigma(\sigma^t)$ where $f_\Sigma(\sigma^t) = \{A \in P(Y) : t \in Cl_T\, f^{-1}(A^-)\}$. Let $\underset{\sim}{M} = \{N_{y_n} : y_n \in C_t, N_{y_n}$ an open nbhd. of y_n, $N_{y_n} \notin f_\Sigma(\sigma^t)\}$. Then

$\underset{\sim}{M} \cup \{Y-C_t\}$ is a countable open cover of Y, and hence, because of countable paracompactness of Y, there exists a l.f. open refinement $\{0_\alpha : \alpha \in A\}$ of $\underset{\sim}{M} \cup \{Y-C_t\}$ which also covers Y. From Prop. 1.26(1), as no element of $\underset{\sim}{M} \cup \{Y-C_t\}$ is in $f_\Sigma(\sigma^t)$, $0_\alpha \notin f_\Sigma(\sigma^t)$ for every α in A. If $\underset{\sim}{G} = \{f^{-1}(0_\alpha) : \alpha \in A\}$, then, by the continuity of f, it follows that $\underset{\sim}{G}$ is a l.f. open cover of X, and hence by (E_1), $\underset{\sim}{G}$ is closure-preserving in T. Thus, $\bigcup_{\alpha\in A} Cl_T\, f^{-1}(0_\alpha) = Cl_T \bigcup_{\alpha\in A} f^{-1}(0_\alpha) = Cl_T X = T$, so that there exists at least one α' in A such that $t \in Cl_T f^{-1}(0'_\alpha)$ i.e. such that $0'_\alpha \in f_\Sigma(\sigma^t)$, a contradiction. That y^t is unique follows as in Th. 3.12.

3.17 __Theorem__ : Let X be a T_1-dense subspace of an R_0-space T and let Y be a countably metacompact T_3-space. If $f : X \to Y$ is a continuous map and if :

(E'_1) : every p.f. open cover of X is closure-preserving in T, and

(E'_2) : for every $t \in T$, there exists a countable closed subset C_t of Y such that $t \in Cl_T\, f^{-1}(C_t)$, and $t \notin Cl_T\, f^{-1}(\overline{Y-C_t})$,

then f has a continuous extension $\overline{f} : T \to Y$ iff for disjoint closed sets F_1, F_2 in Y, $Cl_T\, f^{-1}(F_1) \cap Cl_T(F_2) = \phi$.

__Proof__ : We first recall that a space Y is countably metacompact iff every countable open cover of Y has a p.f. open refinement that also covers Y. Hence the result follows directly from the proof of Th. 3.16, when we replace the terms 'l.f.' and 'countably paracompact' by the terms 'p.f.' and 'countably metacompact' respectively.

Finally, we close this chapter by proving a result which generalizes Theorem D of Blefko [6].

3.18 Theorem : Let X be dense in a first-countable R_o-space T and let Y be a T_3-space. Then a continuous map $f : X \to Y$ has a continuous extension $\overline{f} : T \to Y$ iff for disjoint closed sets F_1, F_2 in Y, $Cl_T\ f^{-1}(F_1) \cap Cl_T\ f^{-1}(F_2) = \phi$.

Proof : We again assign the LO-proximity δ_o to T and Y and let X have the derived subspace proximity δ_X from (T, δ_o). Then the necessity of the condition follows from Lemma 3.1. For the sufficiency, as in the previous theorems, it is enough to prove that, for each $t \in T$, $f_\Sigma(\sigma^t)$ converges to a unique y^t in T. Let $t \in T$. Then as X is dense in T and as T is first countable, there exists a sequence $\langle x_n \rangle$ in X which converges to t in T. We will now show that the corresponding sequence $\langle f(x_n) \rangle$ converges to some point y^t of Y. If $\langle f(x_n) \rangle$ is the constant sequence, then the result follows trivially by setting $y^t = f(x_n)$. If $\langle f(x_n) \rangle$ is not a constant sequence, and if it also does not converge to some point y^t of Y, then we can select a subsequence $\langle f(x_{n,i}) \rangle$ of $\langle f(x_n) \rangle$ such that all its elements are distinct.

Let $F_1 = \{f(x_{n,1}), f(x_{n,3}), f(x_{n,5}) \ldots \}$ and

$$F_2 = \{f(x_{n,2}), f(x_{n,4}), f(x_{n,6}) \ldots \} .$$

As $\langle f(x_n) \rangle$ is not convergent, F_1, F_2 are closed in Y, and since $F_1 \cap F_2 = \phi$, we must have that $Cl_T\ f^{-1}(F_1) \cap Cl_T\ f^{-1}(F_2) = \phi$ which is a contradiction as $\langle x_n \rangle$ converging to t implies that $t \in Cl_T\ f^{-1}(F_1) \cap Cl_T\ f^{-1}(F_2)$. Hence, $\langle f(x_n) \rangle$ converges to some point y^t in Y, which means that the sequence $\langle x_n \rangle$ is enventually in

$f^{-1}(N_{y^t})$ for every nbhd. N_{y^t} of y^t in Y. Combining this with the result that $\langle x_n \rangle$ converges to t in T, we get that, for every nbhd. N_t, N_{y^t} of t, y^t in T and Y respectively, the net $\langle x_n \rangle$ is eventually in $N_t \cap f^{-1}(N_{y^t})$ and hence $t \varepsilon \mathrm{Cl}_T\, f^{-1}(N_{y^t})$, i.e. $N_{y^t} \varepsilon f_\Sigma(\sigma^t)$. Therefore $f_\Sigma(\sigma^t)$ converges to y^t. The proof for the uniqueness of y^t is as in Th. 3.12.

CHAPTER 4

WALLMAN COMPACTIFICATIONS

We begin the present chapter by showing that to every Wallman type compactification of a topological space X, there corresponds a Smirnov-type compactification of X. In particular, if $\mathcal{L}$ is a normal[separating] base on a Tychonoff $[T_1$-]space X, then we will prove that $\mathcal{L}$ induces a natural EF-[LO-] proximity $\delta(\mathcal{L})$ on X and that the Wallman compactification $W(X,\mathcal{L})$ is, in fact, homeomorphic to the family of all clusters [bunches generated by $\mathcal{L}$-ultrafilter] over $(X,\delta(\mathcal{L}))$ with the A-topology τ_A. We will also obtain a necessary and sufficient condition for a Hausdorff compactification of a Tychonoff space to be Wallman, and show that our result includes those of Banashewski [5], Alo and Shapiro [1], Njåstad [27], and Steiner [35]. In the final section, we will apply the results of chapter 2 to obtain the extension theorem of Ponomarev which states that every continuous map $f : X \to Y$, X, Y being T_1 spaces, has a continuous extension $wf : wX \to wY$, where wX, wY are the Wallman compactifications of X and Y respectively.

4.1 <u>Lemma</u> : If $\mathcal{L}$ is a separating base on a T_1-space (X,τ), then $\delta = \delta(\mathcal{L})$ defined by :

$A \not\delta B$ iff there exist L_A, L_B in $\mathcal{L}$ such that

$A \subset L_A$, $B \subset L_B$ and $L_A \cap L_B = \emptyset$,

is a compatible separated LO-proximity on X. Further, if $\mathcal{L}$ is a normal base, then $\delta(\mathcal{L})$ is a separated EF-proximity on X.

<u>Proof</u> : Let $\mathcal{L}$ be a separating base on a T_1-space X ,and $\delta = \delta(\mathcal{L})$ be defined as above. Then :

(i) $A \delta B$ implies $B \delta A$ is self-evident, i.e. (P.1) holds.

(ii) $A \not\delta C$, $B \not\delta C$ implies there exist L_A, L'_C, L_B, L''_C in $\mathcal{L}$ such that $A \subset L_A$, $B \subset L_B$, $C \subset L'_C \cap L''_C$ and $L_A \cap L'_C = \phi = L_B \cap L''_C$. Hence, $(A \cup B) \subset L_A \cup L_B$, $C \subset L'_C \cap L''_C$ and $(L_A \cup L_B) \cap (L'_C \cap L''_C) = \phi$. Since $\mathcal{L}$ is a ring, this implies that $(A \cup B) \not\delta C$. Conversely, $(A \cup B) \not\delta C$ implies there exist $L_{A \cup B}$, L_C in $\mathcal{L}$ such that $A, B \subset A \cup B \subset L_{A \cup B}$, $C \subset L_C$ and $L_{A \cup B} \cap L_C = \phi$. Hence $A \not\delta C$, $B \not\delta C$, thus verifying (P.2).

(iii) If $A \in P(X)$, then $A \subset X$, $\phi \subset \phi$, and $X \cap \phi = \phi$. Also, by Remark 1.32(2), $X, \phi \in \mathcal{L}$ Hence $A \not\delta \phi$, i.e. (P.3) holds.

(iv) (P.4) is obvious from the definition of δ.

(v) To check (P.5), we note that $x \delta y$ implies that if $x \in L_x$, $y \in L_y$, L_x, L_y in $\mathcal{L}$, then $L_x \cap L_y \neq \phi$. As $\mathcal{L}$ is a base for closed sets, this in turn implies that $\overline{x} \cap \overline{y} \neq \phi$, and as X is T_1 we get $x = y$.

(vi) If $C \in P(X)$ and $L_C \in \mathcal{L}$ such that $C \subset L_C$, then, $b \delta C$ for each b in B implies $b \in L_C$ for each b in B, i.e. $B \subset L_C$. But since $A \delta B$, $A \subset L_A$, L_A in $\mathcal{L}$ implies that $L_A \cap L_C \neq \phi$. Hence $A \delta C$, so (P.6') holds. Hence δ is a separated LO-proximity on (X, τ). To show that $\tau = \tau(\delta)$, we observe that $x \delta A$ iff $x \in L_x \in \mathcal{L}$, $A \subset L_A \in \mathcal{L}$ implies $L_x \cap L_A \neq \phi$ iff $\overline{x} \cap L_A \neq \phi$ iff $x \in L_A$ for each $L_A \supset A$, iff $x \in A^-$ (where $^-$ is the τ-closure). Finally, if $\mathcal{L}$ is a normal base, then, in view of Remark 1.32(1), in order to show that $\delta(\mathcal{L})$ is a separated EF-proximity, we need only check the axiom (P.6). Now, $A \not\delta B$ implies there exist L_A, L_B in $\mathcal{L}$ such that $A \subset L_A$, $B \subset L_B$ and $L_A \cap L_B = \phi$. From the normality condition (N) of $\mathcal{L}$, there exist L'_A, L'_B in $\mathcal{L}$ such that $L_A \subset X - L'_A$, $L_B \subset X - L'_B$ and $L'_A \cup L'_B = X$. Clearly, $A \not\delta L'_A$, $B \not\delta L'_B$. Thus, the axiom (P.6) is satisfied.

4.2 Remark : We note here that the normal base $\mathcal{L}$ is an EF-proximity base for $\delta = \delta(\mathcal{L})$ in the sense of Njåstad [27].

In the subsequent work, the following Lemma will be very useful.

4.3 Lemma : If X is a T_1-space and $\mathcal{L}$ a separating base on X, then, for $\mathcal{B} \subset W(X,\mathcal{L})$, $\mathcal{B}$ is closed in the W-topology iff $\mathcal{B} = Cl(\mathcal{B})$ as defined in 2.4 .

Proof : Since the W-closure of $\mathcal{B}$ is given by $\overline{\mathcal{B}} = \cap \{L^* : \mathcal{B} \subset L^*, L \in \mathcal{L}\}$ (see Def. 1.35), $\underset{\sim}{F} \in \overline{\mathcal{B}}$ iff $\underset{\sim}{F} \in L^*$ for each L^* containing $\mathcal{B}$ iff $L \in \underset{\sim}{F}$ whenever $\mathcal{B} \subset L^*$ iff $L \in \underset{\sim}{F}$ whenever L absorbs $\mathcal{B}$ iff $\underset{\sim}{F} \in Cl(\mathcal{B})$.

4.4 Theorem : Let (X,τ) be a T_1-space and $\mathcal{L}$ a separating base on X. Then the Wallman compactification $W(\mathcal{L})$ of X is homeomorphic to the space of all bunches generated by $\mathcal{L}$-ultrafilters over $(X,\delta(\mathcal{L}))$ with the A-topology.

Proof : Let Σ_X be the family of all bunches over $(X,\delta(\mathcal{L}))$ and let $\Sigma \subset \Sigma_X$ be the family of all bunches which are generated by $\mathcal{L}$-ultrafilters on X. (see Prop. 1.28). Let $\xi : W(\mathcal{L}) \to \Sigma \subset \Sigma_X$ be a map defined by $\xi(\underset{\sim}{F}) = b(\underset{\sim}{F})$, the bunch generated by $\underset{\sim}{F}$, $\underset{\sim}{F}$ in $W(\mathcal{L})$. Then, clearly ξ is well-defined and onto Σ. Also, $\underset{\sim}{F} \neq \underset{\sim}{G}$, $\underset{\sim}{F}, \underset{\sim}{G} \in W(\mathcal{L})$ means $b(\underset{\sim}{F}) \neq b(\underset{\sim}{G})$ and hence ξ is also one-to-one. To show that ξ is a homeomorphism, we need only prove that it is also continuous and closed. But this follows from the fact that $\underset{\sim}{F} \in \overline{\mathcal{B}}$ ($\overline{\mathcal{B}}$ closed subset of $W(\mathcal{L})$) iff every $\overline{E}$ in X which absorbs $\mathcal{B}$ is in $\underset{\sim}{F}$(see Lemma 4.3) iff every $E \in P(X)$ which absorbs $b(\mathcal{B})$ is in $b(\underset{\sim}{F})$ iff $b(\underset{\sim}{F}) \in \overline{\xi(\mathcal{B})}$.

4.5 Lemma : Let $\mathcal{L}$ be a separating base on a T_1-space X, and let $\delta = \delta(\mathcal{L})$ be the corresponding LO-proximity on X(see Lemma 4.1). If $\underset{\sim}{F}$ is an

$\mathcal{L}$ -ultrafilter on X, then $\sigma(\underset{\sim}{F}) = \{A \in P(X) : A \,\delta\, F$ for every F in $\underset{\sim}{F}\}$ is a cluster over (X,δ), (called the <u>cluster generated by</u> $\underset{\sim}{F}$)

<u>Proof</u> : (i) $A,B \in \sigma(\underset{\sim}{F})$, $A \not\delta B$ implies there exist L_A, L_B in $\mathcal{L}$ such that $A \subset L_A$, $B \subset L_B$, $L_A \cap L_B = \emptyset$. But $A \,\delta\, F$ for every F in $\underset{\sim}{F}$ implies, by Prop. 1.14, that $L_A \,\delta\, F$ for each F in $\underset{\sim}{F}$, i.e. $L_A \cap F \neq \emptyset$ for each F in $\underset{\sim}{F}$, and hence, as $\underset{\sim}{F}$ is an $\mathcal{L}$-ultrafilter, $L_A \in \underset{\sim}{F}$. Similarly, $L_B \in \underset{\sim}{F}$ which means that $L_A \cap L_B \neq \emptyset$, a contradiction. So (B.1) holds for $\sigma(\underset{\sim}{F})$.

(ii) $A \notin \sigma(\underset{\sim}{F})$, $B \notin \sigma(\underset{\sim}{F})$ implies there exist F_1, F_2 in $\underset{\sim}{F}$ such that $A \not\delta F_1$, $B \not\delta F_2$, implies there exist L_A, L_B, L_{F_1}, L_{F_2} in $\mathcal{L}$ such that

$(A \cup B) \subset L_A \cup L_B$, $F_1 \cap F_2 \subset L_{F_1} \cap L_{F_2}$ and $(L_A \cup L_B) \cap (L_{F_1} \cap L_{F_2}) = \emptyset$,

and as $\mathcal{L}$ is a ring, it follows that $(A \cup B) \not\delta (F_1 \cap F_2)$. But $\underset{\sim}{F}$ is an $\mathcal{L}$-ultrafilter and hence $F_1 \cap F_2 \in \underset{\sim}{F}$, so that $(A \cup B) \notin \sigma(\underset{\sim}{F})$. Conversely, $(A \cup B) \notin \sigma(\underset{\sim}{F})$ implies there exists an F in $\underset{\sim}{F}$ such that $(A \cup B) \not\delta F$ and hence, by (P.2), $A \not\delta F$, $B \not\delta F$, i.e. $A \notin \sigma(\underset{\sim}{F})$ and $B \notin \sigma(\underset{\sim}{F})$, thus verifying (B.2).

(iii) $E \,\delta\, A$ for every $A \in \sigma(\underset{\sim}{F})$ implies $A \,\delta\, F$ for every F in $\underset{\sim}{F}$ implies $E \in \sigma(\underset{\sim}{F})$ and so (B.3') follows.

4.6 <u>Lemma</u> : Let $\mathcal{L}$ be a normal base on X and let $\delta = \delta(\mathcal{L})$. If $\underset{\sim}{F}$ is a prime $\mathcal{L}$-filter on X, then $\sigma(\underset{\sim}{F}) = \{A \in P(X) : A \,\delta\, F$ for each F in $\underset{\sim}{F}\}$ is a cluster over (X,δ). Conversely, given a cluster σ over (X,δ), there exists a unique $\mathcal{L}$-ultrafilter $\sigma \cap \mathcal{L}$ on X which generates σ.

Proof : As (B.2), (B.3) follows exactly as in Lemma 4.5, we need only check (B.1), i.e. A, B in $\sigma(\underset{\sim}{F})$ implies $A\ \delta\ B$. Suppose not. Then there exist L_A, L_B in $\mathcal{L}$ such that $A \subset L_A$, $B \subset L_B$ and $L_A \cap L_B = \emptyset$. As $\mathcal{L}$ is normal, from condition (N), there exist L'_A, L'_B in $\mathcal{L}$ such that $L_A \subset X - L'_A$, $L_B \subset X - L'_B$ and $L'_A \cup L'_B = X$. Clearly, $A \not\delta L'_A$ and hence $L'_A \notin \underset{\sim}{F}$. Similarly, $L'_B \notin \underset{\sim}{F}$, which implies that $L'_A \cup L'_B \notin F$ (since $\underset{\sim}{F}$ is prime), a contradiction. Thus $\sigma(\underset{\sim}{F})$ is a cluster. Conversely, if σ is a cluster over (X,δ), then, as the conditions of Lemma 2.1 are trivially satisfied, there exists a prime $\mathcal{L}$-filter $\underset{\sim}{F} \subset \sigma$. Clearly, $\underset{\sim}{F} \subset \sigma \subset \sigma(\underset{\sim}{F})$, and hence $\sigma = \sigma(\underset{\sim}{F})$. If $\underset{\sim}{G}$ is the unique $\mathcal{L}$-ultrafilter containing $\underset{\sim}{F}$, then $\sigma(\underset{\sim}{F}) \subset \sigma(\underset{\sim}{G})$ so that $\sigma(\underset{\sim}{F}) = \sigma(\underset{\sim}{G})$. Finally, as $L_1, L_2 \in (\sigma \wedge \mathcal{L})$ iff $L_1 \cap L_2 \neq \emptyset$, $\underset{\sim}{G} = \sigma \cap \mathcal{L}$ and hence the uniqueness of $(\sigma \cap \mathcal{L})$ follows.

4.7 Theorem : Let X be a Tychonoff space and $\mathcal{L}$ a normal base on X. If $\delta = \delta(\mathcal{L})$ then the Wallman compactification $W(\mathcal{L})$ of X is homeomorphic to the δ-Smirnov compactification X.

Proof : As proved in Th. 4.4, if Σ denotes the family of all bunches over (X,δ) which are generated by $\mathcal{L}$-ultrafilters, then $\xi : W(\mathcal{L}) \to \Sigma$ defined by, for $\underset{\sim}{F} \in W(\mathcal{L})$, $\xi(\underset{\sim}{F}) = b(\underset{\sim}{F})$ is a homeomorphism onto Σ. Also, as (X,δ) is an EF-space (Lemma 4.1), by Th. 2.13, the map $\theta = \theta_X : (\Sigma, \tau_A) \to \underset{\sim}{X}$ given by $\theta(b(\underset{\sim}{F})) = \sigma_\theta$, the unique cluster containing $b(\underset{\sim}{F})$, is continuous. But from Lemma 4.6 and the fact that $\underset{\sim}{F} \subset b(\underset{\sim}{F}) \subset \sigma_\theta$, it is clear that $\sigma_\theta = \sigma(\underset{\sim}{F})$. Hence, $b(\underset{\sim}{F}) \neq b(\underset{\sim}{G})$ implies $\sigma(\underset{\sim}{F}) \neq \sigma(\underset{\sim}{G})$, i.e. θ is one-to-one. Finally, by making use of Lemma 4.6 once again, we note that θ is also onto X. Hence, the map $\theta \circ \xi : W(\mathcal{L}) \to \underset{\sim}{X}$ too is continuous, one-to-one and onto X. Since $W(\mathcal{L})$ is compact and $\underset{\sim}{X}$ is Hausdorff, it follows that $\theta \circ \xi$ is a homeomorphism (Bourbaki [7]).

We now use the above theorem, in conjunction with a few well-known properties of EF-spaces, to obtain simpler proofs of some of the recent results in Wallman compactifications. We will first recall the following definition :

4.8 Definition : (Steiner and Steiner [32]) Let $\mathcal{L}, \mathcal{L}'$ be two normal bases on X. Then $\mathcal{L}$ is said to separate $\mathcal{L}'$ iff L_1', L_2' in $\mathcal{L}'$, $L_1' \cap L_2' = \phi$, implies that there exist L_1, L_2 in $\mathcal{L}$ such that $L_1' \subset L_1$, $L_2' \subset L_2$ and $L_1 \cap L_2 = \phi$.

4.9 Remark : It is clear that $\mathcal{L}$ separates $\mathcal{L}'$ iff $\delta(\mathcal{L}') < \delta(\mathcal{L})$. Also, we recall that for EF-spaces, there exists one-to-one order isomorphism between the EF-proximities on X and the corresponding Smirnov compactifications of X (see Smirnov [31]).

4.10 Theorem : (Steiner and Steiner [32]). If $\mathcal{L}, \mathcal{L}'$ are two normal bases on X. Then $W(\mathcal{L}') \leq W(\mathcal{L})$ iff $\mathcal{L}$ separates $\mathcal{L}'$.

Proof : In view of Th. 4.7, $W(\mathcal{L}') \leq W(\mathcal{L})$ iff the $\delta(\mathcal{L}')$-Smirnov compactification of X $\leq$ $\delta(\mathcal{L})$-Smirnov compactification of X iff $\delta(\mathcal{L}') \leq \delta(\mathcal{L})$ (from Remark 4.9) iff $\mathcal{L}$ separates $\mathcal{L}'$.

4.11 Corollary : (Steiner [35]). $W(\mathcal{L})$ is equivalent to $W(\mathcal{L}')$ iff $\mathcal{L}$ and $\mathcal{L}'$ mutually separate each other.

We will now obtain a necessary and sufficient condition for a Hausdorff compactification of X to be Wallman, a motivation for which is provided by the known behaviour of the closures in $W(\mathcal{L})$ of members of $\mathcal{L}$ (Prop. 1.48).

4.12 Theorem : A necessary and sufficient condition for a Hausdorff compactification Y of X to be Wallman is that X has a normal base $\mathcal{L}$ such that :

(i) $Cl_Y(L_1 \cap L_2) = Cl_Y(L_1) \cap Cl_Y(L_2)$, L_1, L_2 in $\mathcal{L}$

and (ii) for $y_1, y_2 \in Y$, $y_1 \neq y_2$, there exist L_{y_i} in $\mathcal{L}$ such that

$$y_i \in Cl(L_{y_i}),\ i = 1,2 \text{ and } L_{y_1} \cap L_{y_2} = \phi .$$

Proof : As the necessity is obvious from Prop. 1.48, we need prove only the sufficiency. Since Y is compact Hausdorff, it has the unique compatible EF-proximity δ_o. Clearly, this induces a subspace EF-proximity δ_X on X. From condition (i), it follows that $\delta_X > \delta(\mathcal{L})$, and hence $Y \geq W(\mathcal{L})$. But this means that there is a continuous function f from Y onto $W(\mathcal{L})$ which is the unique extension of the identity map on X. As Y is compact and $W(\mathcal{L})$ is Hausdorff, to show that f is a homeomorphism it is enough to show that f is one-to-one. Let $y_1, y_2 \in Y$, $y_1 \neq y_2$. Then, from (ii), there exist L_i in $\mathcal{L}$ such that $y_i \in Cl_Y(L_i)$, $i = 1,2$ and $L_1 \cap L_2 = \phi$. As f is continuous, it follows that $f(y_i) \in Cl_{W(\mathcal{L})}(L_i)$, $i = 1,2$ where $L_1 \cap L_2 = \phi$. But $Cl_{W(\mathcal{L})}(L_i) = \{ \underset{\sim}{F} \in W(\mathcal{L}) : L_i \in \underset{\sim}{F} \}$ (see 1.35). Hence, $L_i \in f(y_i)$, $i = 1,2$. This clearly means that, as $L_1 \cap L_2 = \phi$, $f(y_1) \neq f(y_2)$ i.e. f is one-to-one.

As mentioned earlier, the above Th. 4.12 includes a number of already known results, which we shall now state as corollaries to Th.4.12.

4.13 Corollary : (Banashewski [5]). A Hausdorff compactification Y of a Tychonoff space X is Wallman iff X has a normal base such that

(i) $Cl_Y(L_1 \cap L_2) = Cl_Y(L_1) \cap Cl_Y(L_2)$, for L_1, L_2 in $\mathcal{L}$

(ii') $\{Cl_Y (L) : L \in \mathcal{L}\}$ is a base for closed sets in Y.

Proof : In view of Th. 4.12, we need only prove that (ii') implies the condition (ii) of Th. 4.12 (since the necessary part is obvious from

Prop. 1.48 (a) and (b)). Let (ii') hold and suppose $y_1, y_2 \in Y$, $y_1 \neq y_2$. As Y is Hausdorff, there exist disjoint open nbhds. N_1, N_2 of y_1, y_2 respectively. From (ii'), for each y in $Y-\{y_1\}$, there exist an L_y in $\mathcal{L}$ such that $y \in (Y-Cl_Y(L_y)) \subset Y-\{y_1\}$. As $(Y-N_1)$ is a compact subset of Y there exist finitely many indicies i such that

$$(Y - N_1) \subset \bigcup_{i=1}^{n} (Y-Cl_Y(L_{y_i})) = Y - \bigcap_{i=1}^{n} Cl_Y(L_{y_i}) \subset Y -\{y_1\} .$$ But from (i) ,

$\bigcap_{i=1}^{n} Cl_Y(L_{y_i}) = Cl_Y \left(\bigcap_{i=1}^{n} L_{y_i} \right)$. Let $L_1 = \bigcap_{i=1}^{n} L_{y_i}$. Then, because $\mathcal{L}$ is a ring, $L_1 \in \mathcal{L}$. Thus, $y_1 \in Cl_Y(L_1) \subset N_1$. Similarly, there exists an L_2 in $\mathcal{L}$ such that $y_2 \in Cl_Y(L_2) \subset N_2$, and as $N_1 \cap N_2 = \phi$, (ii) follows.

4.14 Corollary : (Alo and Shapiro [1]). A Hausdorff compactification Y of a Tychonoff space X is Wallman iff

(i) $Cl_Y(L_1 \cap L_2) = Cl_Y(L_1) \cap Cl_Y(L_2)$, for L_1, L_2 in $\mathcal{L}$

(ii") For each $y \in Y$ and for each nbhd. N of y , there exists an L in $\mathcal{L}$ such that $y \in Cl_Y(L) \subset N$.

Proof : As before, we need only show that (ii") implies (ii) of Th.4.12. But this is immediate from the fact that Y is Hausdorff and hence every pair of distinct points have disjoint nbhds. in Y.

4.15 Corollary : (Njåstad [27]). A Hausdorff compactification Y of a Tychonoff space X is Wallman iff the corresponding proximity has a productive base of closed sets.

Proof : Alo and Shapiro [1] have proved that if Y is a Hausdorff compactification of X and if the corresponding proximity has a productive base of closed sets, then condition (ii") of Corollary 4.14 holds, and hence, as proved in the above mentioned Corollary, condition (ii) of

of Th. 4.12 also holds. The converse is obvious from Remark 4.2.

In obtaining the last corollary to Th. 4.12, we shall make use of the following definition.

4.16 <u>Definition</u> : A family $\hat{\mathcal{L}}$ of closed sets in a topological space Y <u>has the trace property w.r.t.</u> a subspace <u>X</u> iff $\{\bigcap_{i=1}^{n} \hat{L}_i : \hat{L}_i \in \hat{\mathcal{L}}\} \neq \phi$ implies that $\bigcap_{i=1}^{n} (\hat{L}_i \cap X) \neq \phi$.

4.17 <u>Corollary</u>: (Steiner [35]). A Hausdorff compactification Y of X is Wallman iff Y has a normal base $\hat{\mathcal{L}}$ with the trace property w.r.t. X.

<u>Proof</u> : Necessity follows from Prop. 1.48 (a) and (e), by setting $\hat{\mathcal{L}} = \{Cl_Y(L) : L \in \mathcal{L}\}$. To prove the sufficiency, we must prove that if Y has such an $\hat{\mathcal{L}}$ then X has a normal base $\mathcal{L}$ which satisfies (i) and (ii) of Th. 4.12. Let $\mathcal{L} = \hat{\mathcal{L}} \cap X = \{L_i : L = \hat{L} \cap X, \hat{L} \in \hat{\mathcal{L}}\}$. We now show that $\mathcal{L}$ is a normal base on X.

(1) $x \in U$, $U = V \cap X$, V open in Y, and $\hat{\mathcal{L}}$ is a base for closed sets in Y implies that there exists an $\hat{L}$ in $\hat{\mathcal{L}}$ such that $x \in (Y-\hat{L}) \subset V$ and hence $x \in (X-(\hat{L} \cap X)) \subset U$, i.e. $\mathcal{L}$ is a base for closed sets in X.

(2) $x \notin F$, $F = K \cap X$, K closed in X, and $\hat{\mathcal{L}}$ is separating implies that there exists an $\hat{L}$ in $\hat{\mathcal{L}}$ such that $x \in \hat{L} \subset (Y-K)$ and hence $x \in (\hat{L} \cap X) \subset X-F$, i.e. $\mathcal{L}$ is separating.

(3) $\hat{L}_1, \hat{L}_2$ in $\hat{\mathcal{L}}$, $L_1 \cap L_2 = \phi$ and $\hat{\mathcal{L}}$ has the trace property w.r.t. X implies that $\hat{L}_1 \cap \hat{L}_2 = \phi$. From normality condition (N) of $\hat{\mathcal{L}}$, there exist $\hat{L}'_1, \hat{L}'_2$ in $\hat{\mathcal{L}}$ such that $\hat{L}_i \subset Y-\hat{L}'_i$, $i = 1,2$ and

$L'_1 \cup L'_2 = Y$. Consequently, $L_i \subset X-L'_i$, $i = 1,2$ and $L'_1 \cup L'_2 = X$. Thus $\mathcal{L}$ is a normal base on X. Finally, since $Cl_Y(L) = \hat{L}$ for each L in $\mathcal{L}$, conditions (i) and (ii) are trivially satisfied by $\mathcal{L}$ (as $\hat{\mathcal{L}}$ is a normal base on Y).

4.18 Remark : As is well-known (see Gillman and Jerison [12]), if $\underset{\sim}{L}$ is a closed ultrafilter on X and if $f : X \to Y$ is a continuous map, then the family $f^{\#}(\underset{\sim}{L}) = \{E \in P(Y) : E \text{ closed in } Y,\ f^{-1}(E) \in \underset{\sim}{L}\}$ is a prime closed filter on Y. We will now prove a stronger result.

4.19 Lemma : If X,Y are topological space and $f: X \to Y$ is a continuous and closed map , then for a closed ultrafilter $\underset{\sim}{L}$ on X, $f^{\#}(\underset{\sim}{L})$ (as defined in Remark 4.18) is a closed ultrafilter on Y.

Proof : Because of Remark 4.18, we need only show that $f^{\#}(\underset{\sim}{L})$ is maximal. We first note that $f(\underset{\sim}{L}) = \{f(L) : L \in \underset{\sim}{L}\} \subset f^{\#}(\underset{\sim}{L})$. This is so because $f(L)$ is closed in Y and $L \subset f^{-1}(f(L))$ for each L in $\underset{\sim}{L}$. To prove that $f^{\#}(\underset{\sim}{L})$ is maximal, we must show that if E is closed in Y and $E \cap M \neq \phi$ for each M in $f^{\#}(\underset{\sim}{L})$, then $E \in f^{\#}(\underset{\sim}{L})$. But $E \cap M \neq \phi$ for each M in $f^{\#}(\underset{\sim}{L})$ and $f(\underset{\sim}{L}) \subset f^{\#}(\underset{\sim}{L})$ implies that $E \cap f(L) \neq \phi$ for each L in $\underset{\sim}{L}$ and hence $f^{-1}(E) \cap L \neq \phi$ for each L in $\underset{\sim}{L}$. Since $\underset{\sim}{L}$ is maximal, $f^{-1}(E) \in \underset{\sim}{L}$ and consequently $E \in f^{\#}(\underset{\sim}{L})$.

4.20 Remark : We note here that if X is a T_1-space and $\mathcal{L}$ is a separating base of all closed sets in X, then $\delta_o = \delta(\mathcal{L})$. This follows from the fact that $A \not\delta_o B$ implies $A^- \cap B^- = \phi$ and as $A^-, B^- \in \mathcal{L}$, this means that $A \not\delta(\mathcal{L}) B$. The converse is true in view of Th. 1.16. Hence, if wX denotes the corresponding Wallman compactification of X, then from Th. 4.4, wX is homeomorphic to $\underset{\sim}{X}$, the space of all bunches generated by closed ultrafilters over (X,δ_o) with the A-topology.

4.21 Theorem : (Ponomarev [30]). Let X, Y be T_1-spaces, and let $f : X \to Y$ be a continuous and closed map. Then f has a continuous extension $wf : wX \to wY$. Further, if f is onto, then so is wf.

Proof : Let X, Y have the separated LO-proximity δ_0. Then, by Prop.1.20, f is p-continuous, and so, by Th. 2.21 f has a continuous extension $f_\Sigma : wX \to \Sigma_Y$ (note that $\underset{\sim}{X}$ is homeomorphic to wX as in Remark 4.20). Also, $f_\Sigma(b(\underset{\sim}{L})) = b(f^{\#}(\underset{\sim}{L}))$ for every closed ultrafilter $\underset{\sim}{L}$ on X. This follows as $f_\Sigma(b(\underset{\sim}{L})) = \{E \in P(Y) : f^{-1}(E) \in b(\underset{\sim}{L})\} = \{E \in P(Y) : f^{-1}(\overline{E}) \in \underset{\sim}{L}\}$ $= \{E \in P(Y) : \overline{E} \in f^{\#}(\underset{\sim}{L})\} = b(f^{\#}(\underset{\sim}{L}))$. Hence, by Lemma 4.19 and Prop.1.28, $f_\Sigma(b(\underset{\sim}{L})) \in wY$., i.e. $f_\Sigma : wX \to wY$. Finally, if f is onto, and if $b(\underset{\sim}{L'})$ is a bunch generated by a closed ultrafilter $\underset{\sim}{L'}$ on Y, then $\{f^{-1}(L') : L' \in \underset{\sim}{L'}\}$ is a closed filter base on X and is hence contained in a closed ultrafilter $\underset{\sim}{L}$ on X. Clearly $\{f(f^{-1}(L')) : L' \in \underset{\sim}{L'}\} \subset \{f(L) : L \in \underset{\sim}{L}\}$ and as f is closed and $\underset{\sim}{L'}$ is maximal, $\underset{\sim}{L'} = f(\underset{\sim}{L})$. Thus, $f_\Sigma(b(\underset{\sim}{L})) = b(\underset{\sim}{L'})$, i.e., f_Σ is onto. Setting $wf = f_\Sigma$, we get the desired result.

CHAPTER 5

WALLMAN REALCOMPACTIFICATIONS

In the previous chapter, we studied some properties of compact spaces and Wallman compactifications. The purpose of the present chapter is to obtain analogous results for realcompact spaces and Wallman realcompactifications. Thus, we will be providing alternate proofs to some known extension results concerned with realcompact spaces, e.g. to the result of Gillman and Jerison which states that if X,Y are Tychonoff, then every continuous map $f : X \to Y$ can be extended continuously to $f_\nu : \nu X \to \nu Y$. We will also obtain a concrete realization of the Hewitt realcompactification νX of a Tychonoff space X as the space of all clusters over (X,δ_F) generated by real Z-ultrafilters with the A-topology. A major portion of this chapter, however, will be devoted to the study of the space $\eta(\mathcal{L})$, corresponding to a c.p. normal base $\mathcal{L}$ on X. We will prove that a necessary and sufficient condition for $\eta(\mathcal{L})$ to be realcompact is that $\bigcap_{n=1}^{\infty} \mathrm{Cl}_{X^Q}(L_n) = \mathrm{Cl}_{X^Q}(\bigcap_{n=1}^{\infty} L_n)$, [where X^Q denotes the Q-closure of X in $W(\mathcal{L})$ and $\{L_n\}$ is a sequence in $\mathcal{L}$] and also show that this result is an improvement over the sufficiency condition of Steiner and Steiner [34]. Further, we will show that $\eta(\mathcal{L})$ is, in fact, homeomorphic to νX iff

$$\mathrm{Cl}_T(\bigcap_{n=1}^{\infty} F_n) = \bigcap_{n=1}^{\infty} \mathrm{Cl}_T(F_n), \text{ where (i) } T = \eta(\mathcal{L}),\ F_n \in Z(X) \text{ and (ii) } T = \nu X,$$

$F_n \in \mathcal{L}$. Finally, we will obtain some necessary and sufficient conditions for an $\mathcal{L}$ *-realcompactification Y of a Tychonoff space X to be homeomorphic to $\eta(\mathcal{L})$. Results in this direction will be motivated by the

known results for Wallman compactifications such as those of Banashewski [5], Njåstad [27] and Steiner [35] mentioned in chapter 4.

We begin this chapter by showing that a result of Blefko [6] and Engelking [9] follows easily by our results of Chapter 2. For this, we shall require the following result of Gillman and Jerison [12].

5.1 Lemma : A Tychonoff space X is realcompact iff every real prime Z-filter in X converges.

5.2 The Theorem of Blefko, Engelking : Let X be a T_1-dense subspace of an R_o-space T and let Y be a Tychonoff realcompact space. Then a continuous map $f : X \to Y$ has a continuous extension $\bar{f} : T \to Y$ iff for every sequence $\{F_n\}$ of closed subsets of Y, $\bigcap_{n=1}^{\infty} F_n = \phi$ implies that $\bigcap_{n=1}^{\infty} Cl_T\, f^{-1}(F_n) = \phi$.

Proof : To show the sufficiency, let T,Y be assigned the separated LO-proximity δ_o and a compatible separated EF-proximity δ respectively. Let X have the derived subspace proximity δ_X from (T,δ_o). Then, in view of Prop. 1.20, the given condition implies that f is p-continuous, and hence, by Th. 3.4, f has a continuous extension $\bar{f} : T \to \underset{\sim}{Y}$. As in of Th. 2.16, in order to show that the extension of f is into Y, it is enough to prove that for each $t \in T$, $f_\Sigma(\sigma^t)$ converges to some unique y^t in Y. For this, we first observe that the family $\underset{\sim}{F}$ of zero-sets in $f_\Sigma(\sigma^t)$ has the c.i.p., for if not, then there exists a sequence $\{Z_n\}$ of zero-sets in $f_\Sigma(\sigma^t)$ such that $\bigcap_{n=1}^{\infty} Z_n = \phi$, and hence, by hypothesis, $\bigcap_{n=1}^{\infty} Cl_T\, f^{-1}(Z_n) = \phi$, a contradiction since $t \in \bigcap_{n=1}^{\infty} Cl_T\, f^{-1}(Z_n)$. Next, we

note that $\underset{\sim}{F}$ obviously satisfies the conditions of Lemma 2.1, and hence, given a $Z_o \in \underset{\sim}{F}$, there exists a prime Z-filter P such that $Z_o \in \underset{\sim}{P} \subset \underset{\sim}{F}$. As $\underset{\sim}{F}$ has the c.i.p., so does $\underset{\sim}{P}$. From Lemma 5.1, since Y is real-compact, $\underset{\sim}{P}$ converges to some point y^t in Y. Uniqueness of y^t follows exactly as in [Th. 3.12. For the necessity of the given condition, we provide a proof which is essentially the same as that given by Blefko [6]. Let $\bar{f} : T \to Y$ be the continuous extension of f and let $\{F_n\}$ be a sequence of closed sets in Y such that $\bigcap_{n=1}^{\infty} \{F_n\} = \phi$. If possible, let $p_o \in \bigcap_{n=1}^{\infty} Cl_T\, f^{-1}(F_n)$. Then $p_o \in \bigcap_{n=1}^{\infty} Cl_T\, \bar{f}^{-1}(F_n) \subseteq \bigcap_{n=1}^{\infty} \bar{f}^{-1}(\bar{F}_n) = \bigcap_{n=1}^{\infty} \bar{f}^{-1}(F_n)$ $= \bar{f}^{-1}(\bigcap_{n=1}^{\infty} F_n) = \phi$, a contradiction.

5.3 Proposition : If X is a Tychonoff space having the compatible EF-proximity $\delta = \delta_F$ and if $\underset{\sim}{F}$ is a prime Z-filter in X, then $\sigma(\underset{\sim}{F}) = \{A \in P(X) : A \,\delta\, F$ for each F in $\underset{\sim}{F}\}$ is a cluster over (X,δ_F).

Proof : The result is immediate from Lemma 4.6 and the fact that if $\mathcal{L} = Z$, then $\delta_F = \delta(\mathcal{L})$. To check the latter statement, as from Th. 1.16 $\delta_F > \delta(\mathcal{L})$, we need only show that $\delta(\mathcal{L}) > \delta_F$ i.e. we must show that $A \not\delta_F B$ implies $A \not\delta(\mathcal{L}) B$. But this is obvious from Prof. 1.38 (ii). Hence $\delta_F = \delta(\mathcal{L})$. Using the above Proposition, we will now obtain a concrete realization of the Hewitt realcompactification νX of a space X.

5.4 Theorem : Let X be a Tychonoff space and let it be assigned the EF-proximity δ_F. Then the Hewitt realcompactification of X is the space of all clusters over (X,δ_F) generated by the real Z-ultrafilters in X, the space being assigned the A-topology τ_A.

Proof : Let $\hat{X}$ be the space of all clusters over (X,δ_F) generated by the real Z-ultrafilters in X with the A-topology τ_A. To prove that $\hat{X}$ is the Hewitt realcompactification of X, because of Th. 1.44, it is sufficient to show that if Y is a realcompact space, then every continuous map $f : X \to Y$ can be continuously extended to $\hat{f} : \hat{X} \to Y$. Let Y be assigned any compatible EF-proximity δ. Then, by Prop. 1.19, f is p-continuous and hence, from Th. 2.11 f has a continuous extension $f_\Sigma : \hat{X} \subset \Sigma_X \to \Sigma_Y$. To show that the extension is into Y, as before, we need only prove that $f_\Sigma(\sigma)$ converges to some unique y^σ in Y for each σ in X. Let $\sigma \in \hat{X}$, i.e. $\sigma = \sigma(\underset{\sim}{L})$ for some real Z-ultrafilter $\underset{\sim}{L}$ in X. Then $f^{\#}(\underset{\sim}{L}) \subset f_\Sigma(\sigma)$ [where $f^{\#}(\underset{\sim}{L})$ is as defined in Remark 4.18]. This follows from the fact that $A \in f^{\#}(\underset{\sim}{L})$ implies $f^{-1}(A) \in \underset{\sim}{L}$ implies $f^{-1}(A) \in \sigma(\underset{\sim}{L})$ implies $A \in f_\Sigma(\sigma)$. Also, $f^{\#}(\underset{\sim}{L})$ is a prime Z-filter in Y(see Remark 4.18) and as $\underset{\sim}{L}$ is real, so is $f^{\#}(\underset{\sim}{L})$. Applying Lemma 5.1 , as Y is realcompact, $f^{\#}(\underset{\sim}{L})$ converges to some point y^σ in Y and hence $f_\Sigma(\sigma)$ converges to y^σ. The proof for the uniqueness of y^σ is the same as in Th. 3.12 for the uniqueness of y^t.

The above Th. 5.4 provides an easy proof of a result due to Gillman and Jerison quoted in McDowell [21]. But first we note that from (i) 6F (3) page 94 and (ii) 8F page 126, of Gillman and Jerison [12], it follows that :

5.5 Lemma : If X is dense in a realcompact space T, then every real prime Z-filter in X converges in T.

5.6 Theorem : Every continuous function f from a Tychonoff space X to a Tychonoff space Y can be continuously extended to $f_\nu : \nu X \to \nu Y$.

Proof : Let X,Y each be assigned the separated EF-proximity δ_F. Then, as in Th. 5.4, if $\sigma \in \nu X$, the family $f_\Sigma(\sigma)$, (which, by Th. 2.11, is a bunch over

$(Y,\delta_F))$, contains a real prime Z-filter in Y. Hence by Lemma 5.5, $f_\Sigma(\sigma)$ converges to some y^σ in νY, and by a reasoning similar to that of Th. 3.12, y^σ is unique. Let $\theta_{\nu Y} : f_\Sigma(\nu X) \to \nu Y$ be defined by $\theta_{\nu Y}(f_\Sigma(\sigma)) = y^\sigma$, the limit of $f_\Sigma(\sigma)$ in νY. Then for $f_\Sigma(\sigma) \in \nu X$, $\underset{\sim}{A} \subset \nu X$ $\theta_{\nu Y}(f_\Sigma(\sigma)) = y^\sigma \notin \overline{\theta_{\nu Y}(\underset{\sim}{A})}$, implies there exist disjoint closed nbhds. N_1, N_2 in νY of y^σ and $\overline{\theta_{\nu Y}(\underset{\sim}{A})}$ respectively (because νY is T_3). Clearly, $N_2 \cap X$ absorbs $\underset{\sim}{A}$ but does not belong to $f_\Sigma(\sigma)$, i.e. $f_\Sigma(\sigma) \notin \overline{\underset{\sim}{A}}$. Hence $\theta_{\nu Y}$ is a continuous map. Setting $f_\nu = \theta_{\nu Y} \circ f_\Sigma$, the result follows.

We will now shift our attention to the study of some general properties of a normal base $\mathcal{L}$ on a space X. Most of these have been proved to hold in the particular case when $\mathcal{L} = Z$. (see Gillman and Jerison [12]).

5.7 <u>Proposition</u> : If $\mathcal{L}$ is a normal base on a Tychonoff space X then the following hold :

(a) Given a nbhd. U of x in X, there exists a nbhd. $L \in \mathcal{L}$ of x such that $L \subset U$.

(b) An $\mathcal{L}$-filter $\underset{\sim}{F}$ converges to a point p in X iff $\underset{\sim}{F}$ contains all $\mathcal{L}$-nbhds. of p.

(c) If $p \in X$ is a cluster point of an $\mathcal{L}$-filter $\underset{\sim}{F}$ on X, then there exists an $\mathcal{L}$-ultrafilter $\underset{\sim}{U}$ which contains $\underset{\sim}{F}$ and converges to p.

(d) If $\underset{\sim}{F}$ is a prime $\mathcal{L}$-filter on X, then the following are equivalent :

(i) p is a cluster point of $\underset{\sim}{F}$

(ii) $\underset{\sim}{F}$ converges to p

(iii) $\{p\} = \bigcap\{F : F \in \underset{\sim}{F}\}$.

Hence every prime $\mathcal{L}$-filter has at most one cluster point.

(e) For $p \in X$, define $A_p = \{L \in \mathcal{L} : p \in L\}$. Then :

(i) p is a cluster point of an $\mathcal{L}$-filter $\underset{\sim}{F}$ iff $\underset{\sim}{F} \subset A_p$.

(ii) A_p is the unique $\mathcal{L}$-ultrafilter converging to p.

(iii) Distinct $\mathcal{L}$-ultrafilters cannot have a common cluster point.

Proof : (a) U be a nbhd. of x in X. Then, as $\mathcal{L}$ is a separating base, there exist L_1, L_2 in $\mathcal{L}$ such that $x \in L_1 \subset X-L_2 \subset U$. Obviously $L_1 \cap L_2 = \phi$, and hence, from condition (N), there exist L_1', L_2' in $\mathcal{L}$ such that $L_1 \subset X-L_1'$, $L_2 \subset X-L'_2$ and $L_1' \cup L_2' = X$. Thus, $x \in L_1 \subset X-L_1' \subset L_2' \subset X-L_2 \subset U$ so that L_2' is the required nbhd. of x which is contained in U.

(b) Immediate from (a).

(c) Let $\underset{\sim}{N}$ be the $\mathcal{L}$-filter of all $\mathcal{L}$-nbhds. of p. (That $\underset{\sim}{N}$ is non-void is clear from the fact that $X \in \mathcal{L}$.) Then $\underset{\sim}{F} \cup \underset{\sim}{N}$ has the f.i.p. and is hence embeddible in an $\mathcal{L}$-ultrafilter $\underset{\sim}{U}$. By (b), $\underset{\sim}{U}$ converges to p.

(d) As (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) always, we have only to show that (i) implies (ii). Let L_1 be any $\mathcal{L}$-nbhd. of p. Since $\mathcal{L}$ is a base for closed sets, there is an L_2 in $\mathcal{L}$ such that $p \in (X-L_2) \subset L_1$. Again, since $L_1 \cup L_2 = X \in \underset{\sim}{F}$, either $L_1 \in \underset{\sim}{F}$ or $L_2 \in \underset{\sim}{F}$ (because $\underset{\sim}{F}$ is prime). But p is a cluster point of $\underset{\sim}{F}$ and hence $L_2 \notin \underset{\sim}{F}$. So $L_1 \in \underset{\sim}{F}$. Applying (b), $\underset{\sim}{F}$ converges to p.

(e) (i) follows easily from the fact that p is a cluster point of $\underset{\sim}{F}$ iff p belong to every member of $\underset{\sim}{F}$.

(ii) That A_p is a $\mathcal{L}$-filter is obvious. To see that it is an $\mathcal{L}$-ultrafilter, we note that $L \in \mathcal{L}$, $L \notin A_p$ implies, by the disjunctive property of $\mathcal{L}$, that there is an L_2 in $\mathcal{L}$ such that $p \in L_2 \subset X-L_1$. Thus

$L_2 \in A_p$ and $L_2 \cap L_1 = \emptyset$. Hence A_p is an $\mathcal{L}$-ultrafilter. Uniqueness of A_p follows from (i).

(iii) This is also immediate from (i).

5.8 <u>Remark</u> : For the remainder of this section, we will frequently identify X with $w(X) \subset W(\mathcal{L})$, where $w : X \to W(\mathcal{L})$ is the Wallman map. (see Remark 1.35).

5.9 <u>Lemma</u> : Let $\mathcal{L}$ be a normal base on a Tychonoff space X, and let $X \subset T \subset W(\mathcal{L})$. Then :

(i) Every point of T is the limit of a unique $\mathcal{L}$-ultrafilter on X.

(ii) The family $\mathcal{L}^{\#} = \{Cl_T(L) : L \in \mathcal{L}\}$ is a base for closed sets in T.

(iii) $Cl_T(L) \cap X = L$ for each L in $\mathcal{L}$.

(iv) $L_1, L_2 \in \mathcal{L}$, $L_1 \cap L_2 = \emptyset$ implies $Cl_T(L_1) \cap Cl_T(L_2) = \emptyset$.

(v) If $p \in Cl_T(L_o)$ for some L_o in $\mathcal{L}$ then there exists an $\mathcal{L}$-ultrafilter which contains L_o and converges to p.

(vi) $Cl_T(L_1 \cap L_2) = Cl_T(L_1) \cap Cl_T(L_2)$.

<u>Proof</u> : (i) Evident as $T \subset W(\mathcal{L})$.

(ii) Let U be an open nbhd. of p in T and suppose $U = V \cap T$, V be open in $W(\mathcal{L})$. Then by Alo and Shapiro's results (see 1.48(b)) there exists an L' in $\mathcal{L}$ such that $p \in (W(\mathcal{L}) - Cl_{W(\mathcal{L})}(L')) \subset V$. Since $Cl_T(L) \subset Cl_{W(\mathcal{L})}(L)$ for each L in $\mathcal{L}$, this implies that $p \in T - Cl_T(L') \subset U$. Hence (ii) holds.

(iii) From 1.48(e), $Cl_{W(\mathcal{L})} \cap X = L$ for each L in $\mathcal{L}$. As $T \subset W(\mathcal{L})$ we get $Cl_T(L) \cap X \subset Cl_{W(\mathcal{L})} \cap X = L$. Since the reverse inclusion is obvious, (iii) follows.

(iv) $L_1 \cap L_2 = \phi \Rightarrow Cl_{W(\mathcal{L})}(L_1) \cap Cl_{W(\mathcal{L})}(L_2) = \phi$ (from 1.48(a)) implies that $Cl_T(L_1) \cap Cl_T(L_2) = \phi$ as $T \subset W(\mathcal{L})$.

(v) Let $\underset{\sim}{N}$ be the $\mathcal{L}^{\#}$-filter of all $\mathcal{L}^{\#}$-nbhds. of p. Let $\underset{\sim}{G}$ be the trace of $\underset{\sim}{N}$ on X. Then, as X is dense in T and $Cl_T(L) \cap X = L$ for each L in $\mathcal{L}$, it follows that $\underset{\sim}{G}$ is an $\mathcal{L}$-filter base on X. Since $p \in Cl_T(L_o)$, using (iv) we get that $\underset{\sim}{G} \cup \{L_o\}$ has the f.i.p. . Therefore there is an $\mathcal{L}$-ultrafilter $\underset{\sim}{U}$ which contains $\underset{\sim}{G} \cup \{L_o\}$ and consequently converges to p.

(vi) As $Cl_T(L_1 \cap L_2) \subset Cl_T(L_1) \cap Cl_T(L_2)$, we need prove only the reverse inclusion. Let $p \in Cl_T(L_1) \cap Cl_T(L_2)$. Then, from (v), there exist $\mathcal{L}$-ultrafilters $\underset{\sim}{F}_1, \underset{\sim}{F}_2$ which contain L_1, L_2 respectively and which converge to p. But from (i), p is the limit of a unique $\mathcal{L}$-ultrafilter $\underset{\sim}{U}$ on X. Hence $\underset{\sim}{F}_1 = \underset{\sim}{F}_2 = \underset{\sim}{U}$. Therefore $L_1, L_2 \in \underset{\sim}{U}$ which means $L_1 \cap L_2 \in \underset{\sim}{U}$ and as $\underset{\sim}{U}$ converges to p, this in turn implies that $p \in Cl_T(L_1 \cap L_2)$.

5.10 <u>Lemma</u> : Let X be a dense subspace of a Tychonoff space Y, and let $\mathcal{L}$ be a normal base on X. If (i) $\{Cl_Y(L) : L \in \mathcal{L}\}$ is a base for closed sets in Y and if for $y_1, y_2 \in Y$, $y_1 \neq y_2$, there exist L_1, L_2 in $\mathcal{L}$ satisfying $y_i \in Cl_Y(L_i)$, $i = 1,2$, and $L_1 \cap L_2 = \phi$ and (ii) $\bigcap_{i=1}^{n} Cl_Y(L_i) = Cl_Y(\bigcap_{i=1}^{n} L_i)$, $L_i \in \mathcal{L}$, then Y is homeomorphic to a subspace of $W(\mathcal{L})$.

<u>Proof</u> : For each $y \in Y$, define $\underset{\sim}{P}_y = \{L \in \mathcal{L} : y \in Cl_Y(L)\}$. Then

(a) $\phi \notin \underset{\sim}{P}_y$, (b) $L_1, L_2 \in \underset{\sim}{P}_y$ implies $y \in Cl_Y(L_1) \cap Cl_Y(L_2) = Cl_Y(L_1 \cap L_2)$ (from (ii)) and hence $L_1 \cap L_2 \in \underset{\sim}{P}_y$. (c) $L_1 \subset L_2 \in \mathcal{L}$, $L_1 \in \underset{\sim}{P}_y$ implies $y \in Cl_Y(L_1) \subset Cl_Y(L_2)$ and hence $L_2 \in \underset{\sim}{P}_y$. Finally (d) $L_1, L_2 \in \mathcal{L}$, $L_1 \cup L_2 \in \underset{\sim}{P}_y$ means $y \in Cl_Y(L_1) \cup Cl_Y(L_2)$ and hence either $L_1 \in \underset{\sim}{P}_y$ or $L_2 \in \underset{\sim}{P}_y$. Thus $\underset{\sim}{P}_y$ is a prime $\mathcal{L}$-filter on X. Let $\underset{\sim}{U}_y \in W(\mathcal{L})$ be the unique $\mathcal{L}$-ultrafilter containing $\underset{\sim}{P}_y$. Define $f : Y \to W(\mathcal{L})$ by $f(y) = \underset{\sim}{U}_y$, for y in Y. Then f is clearly well-defined. Also, if $y_1, y_2 \in Y$, $y_1 \neq y_2$, then from (i), there exist L_i in $\mathcal{L}$ such that $y_i \in Cl_Y(L_i)$, $i = 1,2$ and $L_1 \cap L_2 = \phi$. But this implies that $\underset{\sim}{U}_{y_1} \neq \underset{\sim}{U}_{y_2}$ i.e. f is one-to-one. Let $y \in Y$, $A \in P(Y)$, and $f(y) \notin \overline{f(A)}$, then there exists an L_1 in $\mathcal{L}$ such that L_1 absorbs $f(A)$, $L_1 \notin f(y)$. This means that $\overline{A} \subset Cl_Y(L_1)$ and $y \notin Cl_Y(L_1)$ and so $y \notin \overline{A}$. Hence f is continuous. To see that f is closed, suppose $\underset{\sim}{U}_y \in f(X) \cap \overline{f(A)}$. Then every L which absorbs $f(A)$ also belongs to $\underset{\sim}{U}_y$, i.e. $y \in Cl_Y(L)$ for each $L \in \mathcal{L}$ such that $A \subset Cl_Y(L)$. If $y \notin A^-$, then, from (i), there exists an L_2 in $\mathcal{L}$ such that $y \in Y - Cl_Y(L_2) \subset Y - A^-$. Consequelty $A \subset Cl_Y(L_2)$ and $y \notin Cl_Y(L_2)$, a contradiction. Hence f is closed, showing that Y is homeomorphic to $f(X) \subset W(\mathcal{L})$.

5.11 <u>Propositinn</u> : Let $\mathcal{L}$ be a countably productive (or c.p.) normal base on X and let $X \subset T \subset W(\mathcal{L})$. Then the following are equivalent:

(i) $\bigcap_{n=1}^{\infty} L_n = \phi$ implies $\bigcap_{n=1}^{\infty} Cl_T(L_n) = \phi$, $L_n \in \mathcal{L}$, $n \in \mathbb{N}$

(ii) $\bigcap_{n=1}^{\infty} Cl_T(L_n) = Cl_T(\bigcap_{n=1}^{\infty} L_n)$, $L_n \in \mathcal{L}$, $n \in \mathbb{N}$.

(iii) Every point of T is the limit of a unique real $\mathcal{L}$-ultrafilter on X.

(iv) $X \subset T \subset \eta(\mathcal{L})$.

<u>Proof</u> : That (iii) $\Longleftrightarrow$ (iv) is obvious by the definition of $\eta(\mathcal{L})$ (see Def.1.46). That (ii) $\Longrightarrow$ (i) is also self-evident.

To show that (i) $\Longrightarrow$ (iii) : From 5.9(i), every $t \in T$ is the limit of a unique $\mathcal{L}$-ultrafilter $\underset{\sim}{F}$ on X. To see that $\underset{\sim}{F}$ is real, let $\{L_n\}$ be a sequence in $\underset{\sim}{F}$. Then $\underset{\sim}{F}$ converges to t implies that

$t \in \bigcap_{n=1}^{\infty} Cl_T(L_n)$ and hence from (i) $\bigcap_{n=1}^{\infty} L_n \neq \phi$. (iii) $\Longrightarrow$ (ii) : Since

$Cl_T(\bigcap_{n=1}^{\infty} L_n) \subset \bigcap_{n=1}^{\infty} Cl_T(L_n)$ is always true, we need check only the reverse inclusion. Let $p \in \bigcap_{n=1}^{\infty} Cl_T(L_n)$. From 5.9 (v), for each $n \in \mathbb{N}$ there exists an $\mathcal{L}$-ultrafilter $\underset{\sim}{F}_n$ which contains L_n and converges to p. From (iii), $\underset{\sim}{F}_n = \underset{\sim}{F}$ for each $n \in \mathbb{N}$, where $\underset{\sim}{F}$ is the unique real $\mathcal{L}$-ultrafilter converging to p. Since $L_n \in \underset{\sim}{F}$ for each $n \in \mathbb{N}$, from Lemma 2 of Alo and Shapiro [3] which states that if $\underset{\sim}{F}$ is a real $\mathcal{L}$-ultrafilter then it is countably productive, it follows that $\bigcap_{n=1}^{\infty} L_n \in \underset{\sim}{F}$ and hence p, being the limit of $\underset{\sim}{F}$, is in $Cl_T(\bigcap_{n=1}^{\infty} L_n)$

We now come to the main results concerning the space $\eta(\mathcal{L})$ corresponding to a c.p. normal base $\mathcal{L}$ on X. In order to find the necessary and sufficient conditions for $\eta(\mathcal{L})$ to be realcompact we will make use of the concept of "Q-closure" which is due to Mrowka [23].

5.12 <u>Definition</u> : The <u>Q-closure</u> of a non-empty subset A of a topological space X is the set of all $p \in X$ such that every G_δ set

containing p intersects A.

5.13 <u>Remark</u> : It is known that the Q-closure of a subset is always realcompact.

5.14 <u>Lemma</u> : (Alo and Shapiro [3]) $X \subset \eta(\mathcal{L}) \subset X^Q \subset W(\mathcal{L})$ (where X^Q denotes the Q-closure of X in $W(\mathcal{L})$).

5.15 <u>Lemma</u> : (Alo and Shapiro [3]) $\eta(\mathcal{L})$ is realcompact iff $\eta(\mathcal{L}) = X^Q$.

Using these results, we get the following theorem :

5.16 <u>Theorem</u> : If $\mathcal{L}$ is a c.p. normal base on X, then $\eta(\mathcal{L})$ is realcompact iff :

$$(R) : \bigcap_{n=1}^{\infty} Cl_{X^Q}(L_n) = Cl_{X^Q}\left(\bigcap_{n=1}^{\infty} L_n\right),\ n \in \mathbb{N},\ L_n \in \mathcal{L}.$$

<u>Proof</u> : If $\mathcal{L}$ satisfies condition (R), then from Prop. 5.11 we get that $X^Q \subset \eta(\mathcal{L})$. In view of Lemma 5.14, this means that $X^Q = \eta(\mathcal{L})$ and hence by Lemma 5.15, $\eta(\mathcal{L})$ is realcompact. Conversely, if $\eta(\mathcal{L})$ is realcompact, then $\eta(\mathcal{L}) = X^Q$ and so once again, by Prop. 5.11, it follows that condition (R) is satisfied.

We will now show that this result is an improvement over that of Steiner and Steiner [34]. We first recall the following definition.

5.17 <u>Definition</u> : A sequence of sets $\{L_n\}$ in $\mathcal{L}$ is a <u>nest</u> iff there is a sequence $\{L'_n\}$ in $\mathcal{L}$ such that $X-L'_{n+1} \subset L_{n+1} \subset X-L'_n \subset L_n$, for each $n \in \mathbb{N}$.

$\mathcal{L}$ is <u>nest generated</u> iff, for each L in $\mathcal{L}$, there is a nest $\{L_n\}$ in $\mathcal{L}$ such that $L = \bigcap_{n=1}^{\infty} L_n$.

5.18 <u>Remark</u> : Steiner and Steiner [34] have shown that if $\mathcal{L}$ is a nest generated c.p. normal base on X then $\eta(\mathcal{L})$ is realcompact. A similar

result has also been obtained by Alo and Shapiro [4]. However, as the following lemma will show, Steiner and Steiner's sufficiency condition implies the condition of Th. 5.16 which being both necessary and sufficient, is an improvement.

5.19 <u>Lemma</u> : If $\mathcal{L}$ is a nest generated c.p. normal base on X then

$\bigcap_{n=1}^{\infty} Cl_{X^Q}(L_n) = Cl_{X^Q}(\bigcap_{n=1}^{\infty} L_n)$, for all L_n in $\mathcal{L}$.

<u>Proof</u> : In view of Prop. 5.11, it is enough to show that $\bigcap_{n=1}^{\infty} L_n = \emptyset$ implies $\bigcap_{n=1}^{\infty} Cl_{X^Q}(L_n) = \emptyset$. Suppose on the contrary that $p \in \bigcap_{n=1}^{\infty} Cl_{X^Q}(L_n)$. Then, from Theorem 2.2 of Steiner and Steiner [34] in which the authors have proved that every nest generated c.p. normal base $\mathcal{L}$ on X is precisely the trace on X of all zero-sets in $W(\mathcal{L})$, it follows that for each L_n in $\mathcal{L}$, $L_n = Z_n \cap X$, where $Cl_{W(\mathcal{L})}L_n \subset Z_n$ for some zero-set Z_n in $W(\mathcal{L})$. If $Z = \bigcap_{n=1}^{\infty} Z_n$, then Z is G_δ set in $W(\mathcal{L})$, and hence, by Prop. 1.38(iv), $Z \in Z(W(\mathcal{L}))$. Therefore $p \in \bigcap_{n=1}^{\infty} Cl_{X^Q}(L_n) \subset \bigcap_{n=1}^{\infty} Z_n = Z$. Finally, as $p \in X^Q$, and Z is a G_δ set in $W(\mathcal{L})$ containing p, $Z \cap X \neq \emptyset$ (see Def. 5.12, i.e. $\bigcap_{n=1}^{\infty} L_n \cap X \neq \emptyset$ and hence $\bigcap_{n=1}^{\infty} L_n \neq \emptyset$.

5.20 <u>Theorem</u> : Let $\mathcal{L}$, $\mathcal{G}$ be two c.p. normal bases on X. Then $\eta(\mathcal{L})$ is homeomorphic to $\eta(\mathcal{G})$ iff $\bigcap_{n=1}^{\infty} Cl_T(F_n) = Cl_T(\bigcap_{n=1}^{\infty} F_n)$ for

(i) $T = \eta(\mathcal{L})$, $F_n \in \mathcal{G}$ and

(ii) $T = \eta(\mathcal{G})$, $F_n \in \mathcal{L}$.

Proof : If $\eta(\mathcal{L})$ is homeomorphic to $\eta(\mathcal{G})$, then the necessity follows directly from Prop. 5.11. To prove the sufficiency we first note that conditions (i) and (ii) imply that $\mathcal{L}$ and $\mathcal{G}$ mutually separate each other. Hence, by Corollary 4.11, $W(\mathcal{L})$ is homeomorphic to $W(\mathcal{G})$, which in turn implies that $\eta(\mathcal{L})$ [respectively $\eta(\mathcal{G})$] is homeomorphic to a subspace of $W(\mathcal{G})$ [respectively $W(\mathcal{L})$]. Hence, as (i) and (ii) hold, by Prop. 5.11, for each $p \in \eta(\mathcal{L})$, there exists a $q \in \eta(\mathcal{G})$ which converges to p. But this means that p also converges to q as the following argument shows. If p does not converge to q, then q is not a cluster point of p (because p is an $\mathcal{L}$-ultrafilter). Hence, there exists an L_1 in p such that $q \notin Cl_{\eta(\mathcal{G})}(L_1)$, which means that there is a G_1 in $\mathcal{G}$ such that $L_1 \subset G_1$ and $G_1 \notin q$, i.e. $G_1 \supset L_1$ and $G_1 \cap G_2 = \emptyset$ for some G_2 in q. But $L_1 \in p$ implies $p \in Cl_{\eta(\mathcal{L})}(L_1) \subset Cl_{\eta(\mathcal{L})}(G_1)$ and also $p \in Cl_{\eta(\mathcal{L})}(G_2)$ (since q converges to p and $G_2 \in q$). Thus, $Cl_{\eta(\mathcal{L})}(G_1) \cap Cl_{\eta(\mathcal{L})}(G_2)$ $= Cl_{\eta(\mathcal{L})}(G_1 \cap G_2) \neq \emptyset$, a contradiction as $G_1 \cap G_2 = \emptyset$. Hence q converges to p iff p converges to q. Define $f : \eta(\mathcal{L}) \to \eta(\mathcal{G})$ by $f(p) = q$ iff q converges to p, p in $\eta(\mathcal{L})$. Then f is well-defined map. Also, as p converges to q iff q converges to p and as the spaces are Hausdorff, f is one-to-one and onto. To show that f is continuous, suppose $p \in \eta(\mathcal{L})$, $A \subset \eta(\mathcal{L})$ and $f(p) = q \notin Cl_{\eta(\mathcal{G})} f(A)$. Then there is a G in $\mathcal{G}$ such that G absorbs $f(A)$ but $G \notin q$. Hence $\overline{A} \subset Cl_{\eta(\mathcal{L})}(G)$ and $p \notin Cl_{\eta(\mathcal{L})}(A)$, showing that f is continuous. By symmetry, f^{-1} is also continuous and thus $\eta(\mathcal{L})$ is homeomorphic to $\eta(\mathcal{G})$.

An immediate corollary to the above result is the following :

as Y is Hausdorff, f is also one-to-one. We now show that f is onto $\eta(\mathcal{L})$. Let $\underset{\sim}{F} \in \eta(\mathcal{L})$. Set $\underset{\sim}{F}^* = \{Cl_Y(F) : F \in \underset{\sim}{F}\}$. Then (a) $\phi \notin \underset{\sim}{F}^*$ (b) $F_1, F_2 \in \underset{\sim}{F}$ implies $(F_1 \cap F_2) \in \underset{\sim}{F}$ implies $Cl_Y(F_1 \cap F_2) = Cl_Y(F_1) \cap Cl_Y(F_2)$ (from (i)) $\in \underset{\sim}{F}^*$. (c) $F_1 \in \underset{\sim}{F}$, $F_1 \subset L \in \mathcal{L}$ implies $L \in \underset{\sim}{F}$ and hence $Cl_Y(L) \in \underset{\sim}{F}^*$. (d) For $L \in \mathcal{L}$, $Cl_Y(L) \cap Cl_Y(F) \neq \phi$ for each F in $\underset{\sim}{F}$ implies $Cl_Y(L \cap F) \neq \phi$ for each $F \in \underset{\sim}{F}$ (From (i)) and hence, as $\underset{\sim}{F}$ is an $\mathcal{L}$-ultrafilter, $L \in \underset{\sim}{F}$, i.e. $Cl_Y(L) \in \underset{\sim}{F}^*$. (e) $L_n \in \mathcal{L}$, $n \in \mathbb{N}$ and $\bigcap_{n=1}^{\infty} Cl_Y(L_n) = \phi$ implies, from (i), that $Cl_Y(\bigcap_{n=1}^{\infty} L_n) = \phi$ and hence $\bigcap_{n=1}^{\infty} L_n = \phi$. But as $\underset{\sim}{F}$ is real, this in turn means that $\{L_n\}_{n=1}^{\infty} \not\subset \underset{\sim}{F}$ and so $\{Cl_Y(L_n)\}_{n=1}^{\infty} \not\subset \underset{\sim}{F}^*$. Thus $\underset{\sim}{F}^*$ is a real $\mathcal{L}^*$-ultrafilter, and since Y is $\mathcal{L}^*$-realcompact, $\underset{\sim}{F}^*$ converges to some point y in Y. Hence $y \in \bigcap_{F \in \underset{\sim}{F}} Cl_Y(F)$. But this means that $\underset{\sim}{F}$ converges to y in Y (see Def. 1.41), and therefore $f(y) = \underset{\sim}{F}$. To show that f is continuous, let $y \in Y$, $A \in P(Y)$, $f(y) = \underset{\sim}{F}_y \notin Cl_{\eta(\mathcal{L})} f(A)$. Then there is an L in $\mathcal{L}$ such that L absorbs $f(A)$ and $L \notin \underset{\sim}{F}$. This means that $A \subset Cl_Y(A) \subset Cl_Y(L)$ and $y \notin Cl_Y(L)$. Hence $y \notin Cl_Y(A)$ so that f is continuous. Finally, we prove that f is closed. Let $y \in Y$ and $A \in P(Y)$. If $f(y) = \underset{\sim}{F}_y \in Cl_{\eta(\mathcal{L})} f(A)$, then $L \in \underset{\sim}{F}_y$ whenever L absorbs $f(A)$ i.e. $y \in Cl_Y(L)$ whenever $A \subset Cl_Y(L)$. Hence, as $\{Cl_Y(L) : L \in \mathcal{L}\}$ is a base for closed sets in Y, it follows that $y \in Cl_Y(A)$ which proves the closedness of f. Hence f is a homeomorphism of Y onto $\eta(\mathcal{L})$.

The corollaries which we shall now derive from the above Th. 5.22 are analogues to the results of Njåstad [27] and Steiner [35] (see 4.15 and 4.17).

5.23 Corollary : An $\mathcal{L}^*$-realcompactification Y of X is homeomorphic to $\eta(\mathcal{L})$ iff :

(i) $Cl_Y(\bigcap_{n=1}^{\infty} L_n) = \bigcap_{n=1}^{\infty} Cl_Y(L_n)$, $L_n \in \mathcal{L}$ and

(ii) there exists a c.p. normal base $\mathcal{A}$ for the closed sets in Y such that $\mathcal{L} = \mathcal{A} \cap X$.

Proof : Clearly the existence of a c.p. normal base $\mathcal{A}$ for closed sets in Y such that $\mathcal{L} = \mathcal{A} \cap X$ implies that $\mathcal{A}$ satisfies condition (ii) of Th. 5.22. Hence sufficiency follows from the same theorem. Conversely, by setting $\mathcal{A} = \{Cl_Y(L) : L \in \mathcal{L}\}$, as Y is homeomorphic to $\eta(\mathcal{L})$, $\mathcal{A}$ is a c.p. normal base on Y, satisfying (i) and (ii).

5.24 Definition : Y is a Wallman realcompactification of a Tychonoff space X iff Y is homeomorphic to $\eta(\mathcal{L})$ for some c.p. normal base $\mathcal{L}$ on X.

5.25 Corollary : Y is a Wallman realcompactification of a Tychonoff space X iff :

(i) Y possesses a c.p.normal base $\mathcal{A}$ with the trace property w.r.t. X and

(ii) Y is $(\mathcal{A} \cap X)^*$-realcompact.

If Y has such an $\mathcal{A}$, then Y is homeomorphic to $\eta(\mathcal{A} \cap X)$.

Proof : If Y is homeomorphic to $\eta(\mathcal{L})$, then, by setting $\mathcal{A} = \{Cl_Y(L)\ \ L \in \mathcal{L}\}$, the necessity follows from Prop. 5.11. Conversely, suppose Y has such an $\mathcal{A}$. Let $\mathcal{L} = \mathcal{A} \cap X$. Then, as proved in Corollary 4.15, $\mathcal{L}$ is a normal base on X, and since $\mathcal{A}$ is c.p., so is $\mathcal{L}$. Also, as $Cl_Y(A \cap X) = A$ for all A in $\mathcal{A}$ and $\mathcal{A}$ is a normal base on Y, conditions (i) and (ii) of Th. 5.22 are satisfied. Hence

Y is homeomorphic to $\eta(\mathcal{L}) = \eta(\mathcal{A} \cap X)$.

The above corollary also gives the following analogue of Steiner's Theorem 4 in [35] as a direct consequence.

5.26 <u>Corollary</u> : If Y possesses a c.p. normal base $\mathcal{A}$ of regular closed sets, then Y is a Wallman realcompactification of each of its dense subspaces X for which Y is $(\mathcal{A} \cap X)^*$-realcompact.

CHAPTER 6

SEPARATION SPACES

As mentioned in the first chapter, the separation proximity spaces (or S-spaces), were introduced independently by various authors at about the same time. However, not much literature is available on the subject. The present chapter is devoted to a general study of this particular class of proximity spaces. The results are mainly motivated by the known results in the theories of EF-spaces and LO-spaces. In particular, we shall prove that every T_1-space has a compatible S-proximity and conversely. The concept of the δ-nbhd. of a subset of X will also be defined. It will be shown that every p-continuous map between two S-spaces is always continuous and a sufficient condition will be obtained for the converse to hold. Finally, we will prove that every S-proximity on X has a compatible (generalized) uniform structure (called the S-uniformity), and study some of the basic properties of the same. We begin the present section by showing that similar to an EF-space or a LO-space, every S-space also has an associated topological structure.

6.1 <u>Definition :</u> Let (X,δ) be an S-space, and let $A \in P(X)$. Define $A^{\delta} = \{x \in X : x \,\delta\, A\}$.

6.2 <u>Proposition :</u> In an S-space (X,δ), the $(^{\delta})$ operator associating with each $A \in P(X)$ the set A^{δ} as defined in 6.1, is a Kuratowski closure operator. Moreover, the induced topology τ denoted by $\tau(\delta)$, is a T_1-topology on X.

<u>Proof :</u> We first check that $(^{\delta})$ is indeed a Kuratowski closure operator on $P(X)$.

(i) If $A = \phi$, then from (P.3), $A^\delta = \{x \in X : x \delta \phi\} = \phi$.

(ii) If $a \in A$, then $A \cap a \neq \phi$, and hence by (P.4), $a \delta A$ so that $a \in A^\delta$. Thus, $A \subset A^\delta$.

(iii) $x \in (A^\delta \cup B^\delta)$ iff $x \delta A$ or $x \delta B$ iff $x \delta (A \cup B)$ (by (P.2)), iff $x \in (A \cup B)^\delta$, i.e. $A^\delta \cup B^\delta = (A \cup B)^\delta$.

(iv) $x \in (A^\delta)^\delta$ implies $x \delta A^\delta$. Also, by definition 6.1, $a \delta A$ for each a in A^δ. Hence, using (P.6''), $x \delta A$ i.e. $x \in A^\delta$. So we get that $(A^\delta)^\delta \subset A^\delta$. As the reverse inclusion is obvious from (ii), $(A^\delta)^\delta = A^\delta$.

Thus, $(^\delta)$ satisfies the four Kuratowski closure axioms and therefore is a Kuratowski closure operator. To see that $\tau = \tau(\delta)$ so induced on X is T_1, we observe that for x,y in X, $y \in \tau(\delta)$-closure $\overline{x}$ of x iff $y \delta x$ iff $y = x$ (P.5). This means that $\{x\} = \overline{x}$ for each x in X and hence $\tau(\delta)$ is T_1.

The converse to the above Proposition 6.2 is also true, as the following result shows.

6.3 <u>Proposition</u> : A T_1-space (X,τ) has at least one compatible S-proximity, namely the δ_0' S-proximity defined by : for $A,B \in P(X)$,

$A \delta_0' B$ iff $(A \cap \overline{B}) \cup (\overline{A} \cap B) \neq \phi$.

<u>Proof</u> : We must first show that δ_0' satisfies the axioms (P.1) to (P.5) and (P.6''). However, as the axioms (P.1), (P.2), (P.3) and (P.4) are the direct consequences of the definition of δ_0' and the properties of closures of subsets in X, we need only prove that (P.5) and (P.6'') are satisfied by δ_0'. To check (P.5), let $x, y \in X$. Then $x \delta_0' y$ implies that $(x \cap \overline{y}) \cup (\overline{x} \cap y) \neq \phi$ and as (X,τ) is T_1, this in turn implies that

Proof : (a) $A \in \tau(\delta_1)$ implies $x \not\delta_1 (X-A)$ for each x in A implies $x \not\delta_2(X-A)$ for each x in A implies $A \in \tau(\delta_2)$. Thus $\tau(\delta_1) \subset \tau(\delta_2)$.
(b) $A \not\delta_o^1 B$ implies $(A \cap Cl_{(X,\tau_1)}B) \cup (Cl_{(X,\tau_1)} A \cap B) = \phi$ and as

$Cl_{(X,\tau_1)} T \supset Cl_{(X,\tau_2)} T$ for each $T \in P(X)$, this means that

$(A \cap Cl_{(X,\tau_2)}B) \cup (Cl_{(X,\tau_2)}A \cap B) = \phi$ and hence $A \not\delta_o^2 B$. Therefore,

$\delta_o^1 < \delta_o^2$.

6.7 Theorem : The δ_o' S-proximity (as defined in 6.3) is the finest S-proximity compatible with any T_1-space (X,τ).

Proof : We must prove that if δ is any other compatible S-proximity on (X,τ), then $A \delta_o' B$ implies $A \delta B$. But $A \delta_o' B$ implies that there is an x in X such that $x \in (A \cap \overline{B}) \cup (\overline{A} \cap B)$. Without loss of generality, we assume that $x \in (A \cap \overline{B})$. Now, $x \in \overline{B}$ implies that $x \delta B$, and as $x \in A$, from Prop.1.14, $A \delta B$.

6.8 Definition : If (X,δ) is an S-space, and $A,B \in P(X)$, then B is a δ-neighbourhood of A (denoted by $A << B$) iff $A \not\delta (X-B)$. If B is not a δ-nbhd. of A, then we write $A <\!|\!< B$.

6.9 Proposition : If (X,δ) is an S-space and the relation '$<<$' on $P(X)$ is as above, then the following are true :

N - (i) $X << X$.

N - (ii) $A << B$ implies $A \subset B$.

N - (iii) $A \subset B$, $E \subset F$ and $B << E$ implies $A << F$.

N - (iv) $A << B_i$, $1 \leq i \leq n$ iff $A << \bigcap_{i=1}^{n} B_i$

N - (v) $A << B$ implies $(X-B) << (X-A)$.

N - (vi) $x \in X$, $A \in P(X)$, $x << A$ implies for every subset B of X, either $x << B$ or there is a y in $X-B$ such that $y << A$.

N - (vii) $x << (X-y)$ iff $x \neq y$.

Proof : N - (i) From (P.3), $X \not\delta \phi$ and hence $X << X$.

N - (ii) $A << B$ implies $A \not\delta (X-B)$ implies $A \cap (X-B) = \phi$ implies $A \subset B$.

N -(iii) Let $A \subset B$, $E \subset F$. Then $A <\!\!|\!\!< F$ implies $A\ \delta (X-F)$ which, by Prop. 1.14, implies that $A\ \delta (X-E)$ and hence, again by Prop. 1.14, this means that $B\ \delta (X-E)$. Hence $B <\!\!|\!\!< E$.

N - (iv) $A << B_i$, $1 \leq i \leq n$ iff $A \not\delta X-B_i$, $1 \leq i \leq n$ iff $A \not\delta \bigcup_{i=1}^{n} (X-B_i)$ (by inductive reasoning on (P.2)) iff $A \not\delta (X - \bigcap_{i=1}^{n} B_i)$ iff $A << \bigcap_{i=1}^{n} B_i$.

N - (V) $A << B$ implies $A \not\delta (X-B)$ implies $(X-B) \not\delta A$ (P.1) implies $(X-B) << (X-A)$.

N - (vi) Let $x << A$, $B \in P(X)$ be such that $x <\!\!|\!\!< B$ and $y <\!\!|\!\!< A$ for each y in $(X-B)$. Then $x\ \delta (X-B)$ and $y\ \delta (X-A)$ for each y in $(X-B)$ and hence, using (P.6'') we get that $x\ \delta (X-A)$, i.e. $x <\!\!|\!\!< A$ a contradiction.

N -(vii) $x << (X-y)$ implies $x \not\delta X-(X-y) = y$, and from (P.4) this means $x \neq y$.

6.10 Remark : Property N - (vi) of the above Prop. can be also expressed as : For $A \in P(X)$ and $x \in X$, $x \not\delta A$ implies that for each $B \in P(X)$, either $x \not\delta B$ or there is a b in B such that $b \not\delta A$.

We will now prove that an S-proximity δ can always be defined on a set X in terms of a relation '$<<$' on $P(X)$ which satisfies the conditions of Prop. 6.9.

6.11 Theorem : If X is a set and '$<<$' is a binary relation defined on $P(X)$ which satisfies N-(i) to N-(vii) of Prop. 6.9, then the binary relation δ on $P(X)$ given by : for $A, B \in P(X)$, $A \not\delta B$ iff $A << (X-B)$ is an S-proximity on X.

Proof : We must show that δ satisfies axioms (P.1) to (P.5) and (P.6'').

(i) $A \not\delta B$ implies $A << (X-B)$ implies $B << (X-A)$ (from N-(v)) implies $B \not\delta A$. Hence (P.1) holds.

(ii) $(A \cup B) \not\delta C$ iff $C \not\delta (A \cup B)$ (from (i)) iff $C << X-(A \cup B) = (X-A) \cap (X-B)$ iff $C << X-A$ and $C << X-B$ (from N-iv), thus verifying (P.2).

(iii) $A \delta B$ implies $A \not<< (X-B)$ which, in view of N- i , implies that $B \neq \phi$. Because of (i), we similarly get $A \neq \phi$. Hence (P.3) is holds.

(iv) $x \neq y$ implies $x << X-y$ (from N-vii) implies $x \not\delta y$. Hence (P.5) holds.

(v) Let $a \delta B$, $b \delta C$ for each b in B. If possible, let $a \not\delta C$; then $a << (X-C)$. From N-(vi), this means that either $a << X-B$ or there exists a b in B such that $b << X-C$, i.e. either $a \not\delta B$ or there is a b in B such that $b \not\delta C$. But both of these possibilities give a contradiction. Hence (P.6'') is verified. (P.4) is obvious.

6.12 Theorem : If δ is a compatible S-proximity of a T_1-space (X,τ), then for $A \in P(X)$, $A^o = \{x \in X : x << A\}$.

Proof : We first note that for an open set G in (X,τ), $x \in G$ iff $x \not\delta (X-G)$ iff $x << G$. Set $B = \{x \in X : x << A\}$. Then clearly, $A^o \subset B \subset A$. To prove that $A^o = B$, it is sufficient to prove that $B \in \tau$, i.e. if $x \in B$, then $x << B$. But $x \in B$ implies $x << A$ implies

either $x << B$ or there is a y in $(X-B)$ such that $y << A$(from N-vi). But $y << A$ implies $y \in B$, a contradiction. Hence $x << B$. Thus $B \in \tau$ and therefore $A^o = B$.

6.13 <u>Remark</u> : It is interesting to note here that in the case of an S-proximity δ which is compatible with a T_1-space (X,τ), the following are not always true :

(i) $A << B$ implies $\overline{A} << B^o$ and

(ii) $A << B$ implies $A << B^o$.

A counter example to both of these is again provided by the S-space $(\mathbb{R},\delta_o')$, with $A = (0,1)$ and $B = \mathbb{R} -(1,2)$.

We will now prove a result which is analogous to Prop. 1.19 and Prop. 1.20.

6.14 <u>Proposition</u> : Let (X,δ_1), (Y,δ_2) be two S-spaces. If $f : (X,\delta_1) \to (Y,\delta_2)$ is p-continuous, then the map $f : (X,\tau(\delta_1)) \to (Y,\tau(\delta_2))$ is continuous. In general, the converse need not hold, but it holds if $\delta_1 = \delta_o'$.

<u>Proof</u> : Let f be p-continuous. To prove that f is continuous, we need prove that for each $A \in P(X)$, if $x \in \overline{A}$, then $f(x) \in \overline{f(A)}$. If $x \in \overline{A}$, then $x \,\delta_1\, A$ so that $f(x) \,\delta_2\, f(A)$. Hence $f(x) \in \overline{f(A)}$. That the converse is not always true can be seen by taking $X = Y = \mathbb{R}$, $\delta_1 = \delta_M$ (see 1.17) and δ_2 = the S-proximity δ_o'. Then the identity map i is continuous. But if $A = (0,1)$, $B = (1,2)$, then $A \,\delta_M\, B$ but $i(A) \,\not\delta_o'\, i(B)$. Hence i is not p-continuous. For the last part we note that if δ_1 = the S-proximity δ_o' and f is a continuous map, then $A \,\delta_1\, B$ implies $(A \cap \overline{B}) \cup (\overline{A} \cap B) \neq \phi$ implies $(f(A) \cap f(\overline{B})) \cup (f(\overline{A}) \cap f(B)) \neq \phi$, and as $f(\overline{A}) \subset \overline{f(A)}$ and $f(\overline{B}) \subset \overline{f(B)}$,

we, get $(f(A) \cap \overline{f(B)}) \cup (\overline{f(A)} \cap f(B)) \neq \emptyset$ and so $f(A)\ \delta_o'\ f(B)$. But $\delta_o' > \delta_2$ (from Th. 6.7), and hence $f(A)\ \delta_2\ f(B)$.

6.15 Theorem : If X is a set, (Y,δ_2) an S-space and if $f : X \to Y$ is a one-to-one map, then the coarsest S-proximity δ_1 which may be assigned to X in order to make the map $f : (X,\delta_1) \to (Y,\delta_2)$ p-continuous is given by : $A\ \delta_1\ B$ iff $f(A)\ \delta_2\ f(B)$, A, B in $P(X)$.

Proof : We must first show that δ_1 so defined is an S-proximity on X. As axioms (P.1) to (P.4) follow directly from the fact that δ_2 is an S-proximity on Y , we need check only the axioms (P.5) and (P.6). Let $x\ \delta_1\ y$. Then $f(x)\ \delta_2\ f(y)$ and as δ_2 satisfies (P.5), $f(x) = f(y)$ Since f is one-to-one, this means that $x = y$ and hence (P.5) holds. Again, let $a\ \delta_1\ B$, $b\ \delta_1\ C$ for each b in B. Then $f(a)\ \delta_2\ f(B)$ and $f(b)\ \delta_2\ f(C)$ for each b in B. Since δ_2 satisfies (P.6''), this means that $f(a)\ \delta_2\ f(C)$ and hence $a\ \delta_1\ C$, thereby verifying (P.6'').

Finally, to show that δ_1 as defined above, is the coarsest S-proximity which will make f p-continuous, let δ' be any other S-proximity such that $f : (X,\delta') \to (Y,\delta_2)$ is p-continuous. Then $A \not\delta_1 B$ implies $f(A) \not\delta_2 f(B)$ and as f is p-continuous, this in turn implies that $f^{-1} f(A) \not\delta' f^{-1} f(B)$. Using Prop. 1.14, as $A \subset f^{-1} f(A)$ and $B \subset f^{-1} f(B)$ we get that $A \not\delta' B$. Hence $\delta' > \delta$ which gives the desired result.

For a non-void set X, it is well-known that to every EF-proximity δ defined on $P(X)$ there corresponds a uniform structure (i.e. a subclass of $P(X \times X)$ which satisfies some particular set of axioms). Mozzochi has constructed a similar structure, which he calls the symmetric generalized uniform structure, and has also shown that to every LO-proximity δ defined on $P(X)$ there corresponds a symmetric generalized

uniform structure. (see Mozzochi [25]). We shall now obtain an analogous result for the S-proximities. We first note the following definition.

6.16 <u>Definition</u> : If X is a non-void set, then a U in $P(X \times X)$ is said to be <u>symmetric</u> iff $U = U^{-1}$. A subfamily $\mathcal{U}$ of $P(X \times X)$ is a <u>symmetric family</u> if each U in $\mathcal{U}$ is symmetric.

6.17 <u>Theorem</u> : Let X be a set and $\mathcal{U}$ a collection of non-void, symmetric members of $P(X \times X)$. Define $\delta = \delta(\mathcal{U})$ on $P(X)$ by :

for $A, B \in P(X)$, $A \delta B$ iff $U[A] \cap B \neq \phi$ for every U in $\mathcal{U}$.

Then δ is an S-proximity on X iff $\mathcal{U}$ satisfies the following axioms :

S - (i) $\bigcap \{U : U \in \mathcal{U}\} = \Delta$

S - (ii) If $A \in P(X)$ and $U, V \in \mathcal{U}$, then there is a W in $\mathcal{U}$ such that $W[A] \subset U[A] \cap V[A]$.

S - (iii) If $x \in X$ and $A \in P(X)$ such that $U[x] \cap A \neq \phi$ for each U in $\mathcal{U}$, then given a $V \in \mathcal{U}$, there is a W in $\mathcal{U}$ and a y in A such that $W[y] \subset V[x]$.

(That is, $\mathcal{U}$ is a <u>compatible</u> S-uniformity with the S-proximity δ iff δ satisfies (S-(i), S-(ii) and S-(iii)).

<u>Proof</u> : Let $\mathcal{U}$ satisfy S-(i), S-(ii) and S-(iii). To prove that δ , as defined above, is an S-proximity on X we must check the axioms (P.1) to (P.5) and (P.6'').

(i) $A \delta B$ implies that $U[A] \cap B \neq \phi$ for each U in $\mathcal{U}$. Hence, if $U \in \mathcal{U}$, there is an a in A and a b in B such that $(a,b) \in U$. But as $U = U^{-1}$, this means that $(b,a) \in U$. Hence $a \in U[b] \cap A$, i.e. $U[B] \cap A \neq \phi$. Thus we get that $B \delta A$, so that (P.1) holds for δ .

(ii) If $C \not\delta A$ and $C \not\delta B$ then there exist U_1, U_2 in $\mathcal{U}$ such that $U_1[C] \cap A = \phi$ and $U_2[C] \cap B = \phi$. But from S-(ii), this means there is a W in $\mathcal{U}$ such that $W[C] \subset U_1[C] \cap U_2[C]$. Clearly, $W[C] \cap (A \cup B) = \phi$ and hence $C \not\delta (A \cup B)$. Conversely if $C \not\delta (A \cup B)$ then there is a U in $\mathcal{U}$ such that $U[C] \cap (A \cup B) = \phi$ and hence $U[C] \cap A = \phi = U[C] \cap B$, i.e. $C \not\delta (A \cup B)$. Thus (P.2) is also satisfied.

(iii) (P.3) is evident from the definition of δ itself.

(iv) If $x \in (A \cap B)$, then clearly as $(x,x) \in U$ for each U in $\mathcal{U}$, $x \in U[A] \cap B$ for each U in $\mathcal{U}$. Hence (P.4) holds.

(v) $x \delta y$ implies $U[x] \cap y \neq \phi$ for each U in $\mathcal{U}$ implies that $y \in \bigcap\{U[x] : U \in \mathcal{U}\}$ implies $y \in \Delta[x]$ (from S-(i)) and hence $(x,y) \in \Delta$, i.e. $x = y$, thus verifying (P.5).

(vi) Let $a \delta B$, $b \delta C$ for each b in B. If possible, let $a \not\delta C$. Hence, there is a U in $\mathcal{U}$ such that $U[a] \cap C = \phi$. But $a \delta B$ implies $V[a] \cap B \neq \phi$ for each V in $\mathcal{U}$ and hence, from S-(iii), there is a W in $\mathcal{U}$ and a b in B such that $W[b] \subset U[a]$. Thus $W[b] \cap C = \phi$ which means $b \not\delta C$, a contradiction. Therefore $a \delta C$, and hence (P.6'') is also valid for δ.

Hence if $\mathcal{U}$ satisfies S-(i), S-(ii) and S-(iii), then δ is an S-proximity. Conversely, let δ, as defined above, be an S-proximity on X. We must now prove that $\mathcal{U}$ satisfies S-(i), S-(ii) and S-(iii).

(i) Let $x \in X$, $U \in \mathcal{U}$. As $x \cap x \neq \phi$, from (P.4) $x \delta x$, which means that $U[x] \cap x \neq \phi$, i.e. $(x,x) \in U$. Hence $\Delta \subset U$ for each

U in $\mathcal{U}$. Conversely, if $(x,y) \varepsilon \cap\{U : U \varepsilon \mathcal{U}\}$, then $y \varepsilon U[x]$ for each U in $\mathcal{U}$ and hence $U[x] \cap y \neq \phi$ for every U in $\mathcal{U}$. This implies that $x \delta y$, and by (P.5), it follows that $x = y$, i.e. $\Delta \supset \cap\{U : U \varepsilon \mathcal{U}\}$. Hence $\Delta = \cap\{U : U \varepsilon \mathcal{U}\}$, so that S-(i) is verified.

(ii) Let $U, V \varepsilon \mathcal{U}$ and $A \varepsilon P(X)$ be such that $W[A] \cap (X-(U[A] \cap V[A])) \neq \phi$ for every W in $\mathcal{U}$. For each W in $\mathcal{U}$ set $B(W) = \{x : x \varepsilon (W[A] - U[A] \cap V[A])$, and let $B = \cup\{B(W) : W \varepsilon \mathcal{U}\}$. We will now prove that $A \delta B$. Suppose not. Then, there is a U_A in $\mathcal{U}$ such that $U_A[A] \cap B = \phi$. But $B(U_A) \subset B$. This means that $U_A[A] \cap B(U_A) = \phi$ or that $U_A[A] \subset U[A] \cap V[A]$, a contradiction. Hence $A \delta B$. Let $B_1 = (B - U[A])$ and $B_2 = (B - V[A])$. Then as $U[A] \cap B_1 = \phi = V[A] \cap B_2$, it is clear that $A \not\delta B_1$ and $A \not\delta B_2$, a contradiction to (P.2) as $B_1 \cup B_2 = B$ and $A \delta B$. Consequently S-(ii) must also hold for $\mathcal{U}$.

(iii) Finally, suppose $x \varepsilon X$ and $A \varepsilon P(X)$ such that $U[x] \cap A \neq \phi$ for each U in $\mathcal{U}$ (i.e. $x \delta A$). Let $V \varepsilon \mathcal{U}$. If possible let, for each y in A and for each W in $\mathcal{U}$, $W[y] \cap (X-V[x]) \neq \phi$. Then $y \delta (X - V[x])$ for each y in A. Since $x \delta A$, from (P.6'') it follows that $x \delta X - V[x]$. But this implies that $V[x] \cap (X-V[x]) \neq \phi$, which is impossible. Hence S-(iii) must also hold.

Thus $\mathcal{U}$ is compatible with an S-proximity δ on X iff $\mathcal{U}$ satisfies S-(i), S-(ii) and S-(iii).

6.18 <u>Definition</u> : Given a non-void set X, a symmetric subfamily $\mathcal{U}$ of $P(X \times X)$ is said to be a <u>separated uniformity</u> (or an <u>S-uniformity</u>) <u>on X</u> iff $\mathcal{U}$ satisfies the axioms S-(i), S-(ii), S-(iii) of Th. 6.17, together with the axiom:

S-(iv') : If $V \in P(X \times X)$, $V = V^{-1}$ and $V \supset U$ for some U in $\mathcal{U}$, then $V \in \mathcal{U}$.

Further, if $\mathcal{U}$ is an S-uniformity of X then the pair $(X,\mathcal{U})$ is said to be an <u>S-uniform space</u> (or an <u>SU</u>-space).

6.19 <u>Proposition</u> : Let $(X,\mathcal{U})$ be an SU-space. Then the operator 'Cl' defined on $P(X)$ by : for $A \in P(X)$, $x \in Cl(A)$ iff $U[x] \cap A \neq \phi$ for every U in $\mathcal{U}$, is a Kuratowski closure operator on $P(X)$.

<u>Proof</u> : To prove this, we check the four Kuratowski closure axioms.

(i) That $Cl(\phi) = \phi$ is obvious.

(ii) Let $x \in A \in P(X)$. From S-(i), $(x,x) \in \bigcap\{U : U \in \mathcal{U}\}$, so that $x \in U[x] \cap A$ for each U in $\mathcal{U}$, i.e. $x \in Cl(A)$. Hence $A \subset Cl(A)$.

(iii) $x \in Cl(A) \cup Cl(B)$ iff $x \in Cl(A)$ or $x \in Cl(B)$ iff either $U[x] \cap A \neq \phi$ for each U in $\mathcal{U}$ or $U[x] \cap B \neq \phi$ for each U in $\mathcal{U}$, iff $U[x] \cap (A \cup B) \neq \phi$ for each U in $\mathcal{U}$ iff $x \in Cl(A \cup B)$. Hence $Cl(A) \cup Cl(B) = Cl(A \cup B)$.

(iv) Let $x \in Cl(Cl(A))$. Then $U[x] \cap Cl(A) \neq \phi$ for each U in $\mathcal{U}$. From S-(iii), if $V \in \mathcal{U}$ then there is a y in $Cl(A)$ and a W in $\mathcal{U}$ such that $W[y] \subset V[x]$. But $y \in Cl(A)$ means that $W[y] \cap A \neq \phi$. Hence $V[x] \cap A \neq \phi$.

Since the same is true for each V in $\mathcal{U}$, it follows that $x \in Cl(A)$ and hence $Cl(Cl(A)) \subset Cl(A)$. As the reverse inclusion follows in view of (ii), we get that $Cl(Cl(A)) = Cl(A)$. Thus the result is proved.

6.20 Definition : The topology induced on $(X,\mathcal{U})$ by the 'Cl' operator as given in Prop. 6.19, is called the S-uniform topology (or the SU-topology) on X, denoted by $\tau = \tau(\mathcal{U})$.

6.21 Proposition : Let $(X,\mathcal{U})$ be an SU-space. Then $A \in \tau(\mathcal{U})$ iff for each $x \in A$ there is a U_x in $\mathcal{U}$ such that $U_x[x] \subset A$.

Proof : $A \in \tau(\mathcal{U})$ and $x \in A$ implies that $x \notin Cl(X-A)$ and hence there is a U_x in $\mathcal{U}$ such that $U_x[x] \cap (X-A) = \phi$ i.e. $U_x[x] \subset A$. Conversely, if for each x in A there is a V_x in $\mathcal{U}$ such that $V_x[x] \subset A$, then $x \notin Cl(X-A)$, so that $(X-A)$ is closed in $\tau(\mathcal{U})$, i.e. $A \in \tau(\mathcal{U})$.

6.22 Remark : If $(X,\mathcal{U})$ is an SU-space, then as $U[x] \cap A \neq \phi$ for each U in $\mathcal{U}$ iff $x\ \delta(\mathcal{U})\ A$, it follows that $\tau(\mathcal{U}) = \tau(\delta(\mathcal{U}))$. Hence, since $\delta(\mathcal{U})$ is an S-proximity on X, from Prop. 6.2, $(X,\tau(\mathcal{U}))$ is a T_1-space.

6.23 Theorem : If $(X,\mathcal{U})$ is an SU-space and $A \in P(X)$, then $A^\circ = \{x \in X : U[x] \subset A$ for some U in $\mathcal{U}\}$.

Proof : Let $B = \{x \in X : U[x] \subset A$ for some U in $\mathcal{U}\}$. Then, clearly, $A^\circ \subset B \subset A$. To prove that $A^\circ = B$, it is sufficient to show that $B \in \tau(\mathcal{U})$ or equivalently, that if $x \in Cl(X-B)$ then $x \notin B$. Let $x \in Cl(X-B)$. If possible, let $x \in B$. Then there is a U_x in $\mathcal{U}$ such that $U_x[x] \subset A$. Also $x \in Cl(X-B)$ implies that $V[x] \cap (X-B) \neq \phi$ for each V in $\mathcal{U}$ and hence from S-(iii), there is a y in $(X-B)$ and a

W in $\mathcal{U}$ such that $W[y] \subset U_x[x] \subset A$. But $W[y] \subset A$ implies that $y \in B$, a contradiction. Therefore B is open in $\tau(\mathcal{U})$ so that $A^\circ = B$.

6.24 Corollary : If $(X,\mathcal{U})$ is an SU-space and if $x \in X$, then $\{U[x] : U \in \mathcal{U}\}$ is a nbhd. basis of x.

Proof : If $x \in X$ and G is an open set containing x then, by Prop. 6.21, there is an U_x in $\mathcal{U}$ such that $U_x[x] \subset G$. By Th. 6.23, $x \in (U[x])^\circ$, i.e. $U[x]$ is a nbhd. of x. Hence the result follows.

6.25 Theorem : If $(X,\mathcal{U})$ is an SU-space and if $A \in P(X)$, then $\overline{A} = \bigcap\{U[A] : U \in \mathcal{U}\}$.

Proof : $x \in \overline{A}$ iff $U[x] \cap A \neq \emptyset$ for each U in $\mathcal{U}$ iff $x \in U^{-1}[A]$ $= U[A]$ for each U in $\mathcal{U}$ iff $x \in \bigcap\{U[A] : U \in \mathcal{U}\}$.

6.26 Definition : A subfamily $\mathcal{B}$ of $P(X \times X)$ is said to be a base for an S-uniformity $\mathcal{U}$ on X iff (i) $\mathcal{B} \subset \mathcal{U}$ and (ii) For each U in $\mathcal{U}$, there is a B in $\mathcal{B}$ such that $B \subset U$.

6.27 Definition: A subfamily $\mathcal{S}$ of $P(X \times X)$ is said to be a subbase for an S-uniformity $\mathcal{U}$ iff the set of all finite intersections of elements of $\mathcal{S}$ is a base for $\mathcal{U}$.

6.28 Lemma : Let $(X,\mathcal{U})$ be an SU-space. If a $V \in P(X \times X)$ is closed in $X \times X$, (with the product topology), then for each x in X, $V[x]$ is closed w.r.t. $\tau(\mathcal{U})$.

Proof : Let $x_0 \in X$. To show that $V[x_0]$ is closed, we must prove that every convergent net $\langle y_d : d \in D\rangle$ in $V[x_0]$ converges to a point of $V[x_0]$. Let $\langle y_d : d \in D\rangle$ be a net in $V[x_0]$, which converges to a point b in X. Then, $\langle (x_0, y_d) : d \in D\rangle$ is a net in V which clearly converges to (x_0,b). But V is closed and hence $(x_0,b) \in V$, i.e. $b \in V[x_0]$, thereby yielding the desired result.

6.29 Theorem : Let $(X,\mathcal{U})$ be an SU-space. If $\mathcal{U}$ has a base consisting of closed sets then $\tau(\mathcal{U})$ is T_3.

Proof : The result follows immediately from Corollary 6.24 and Lemma 6.28.

6.30 Theorem : A subfamily $\mathcal{B}$ of $P(X \times X)$ is a base for an S-uniformity $\mathcal{U}$ on X iff $\mathcal{B}$ is a symmetric family satisfying S-(i), S-(ii) and S-(iii).

Proof : The necessity follows from the fact that $\mathcal{B} \subset \mathcal{U}$. To prove the sufficiency, let $\mathcal{B}$ be a symmetric family satisfying the axioms S-(i), S-(ii) and S-(iii). Let $\mathcal{U} = \{ U : U = U^{-1}$ and $U \supset B$ for some B in $\mathcal{B} \}$. Then we will show that $\mathcal{U}$ is an S-uniformity on X. Since by definition itself, $\mathcal{U}$ is a symmetric family which satisfies axiom S-(iv'), we need only check S-(i) to S-(iii).

(i) As $\bigcap\{B : B \in \mathcal{B}\} = \Delta$, S-(i) obviously holds for $\mathcal{U}$.

(ii) Let $A \in P(X)$ and $U_1, U_2 \in \mathcal{U}$. Then there exist B_1, B_2 in $\mathcal{B}$ such that $B_i \subset U_i$, $i = 1,2$. By S-(ii), there is a B_3 in $\mathcal{B}$ such that $B_3[A] \subset B_1[A] \cap B_2[A]$. Hence $B_3[A] \subset U_1[A] \cap U_2[A]$ and as $B_3 \in \mathcal{U}$, we get S-(ii).

(iii) Let $x \in X$, and $A \in P(X)$ be such that $U[x] \cap A \neq \phi$ for each U in $\mathcal{U}$. Hence $B[x] \cap A \neq \phi$ for each B in $\mathcal{B}$ too (since $\mathcal{B} \subset \mathcal{U}$), Also, if $U_1 \in \mathcal{U}$, then there is a B_1 in $\mathcal{B}$ such that $B_1 \subset U_1$. Since $\mathcal{B}$ satisfies S-(iii), there is a B_2 in $\mathcal{B}$ and a y in A such that $B_2[y] \subset B_1[x] \subset U_1[x]$. As $B_2 \in \mathcal{U}$, this means that $\mathcal{U}$ satisfies S-(iii). Consequently $\mathcal{U}$ is an S-uniformity on X.

6.31 Theorem : Let (X,δ) be an S-space. Then there exists an S-uniformity $\mathcal{U}_1(\delta)$ on X such that $\delta(\mathcal{U}_1(\delta)) = \delta$.

Proof : Let A,B be non-void members of $P(X)$. Set $U_{A,B} = ((X \times X) - (A \times B) \cup (B \times A))$. Define $\mathcal{B} = \{U_{A,B} : A \not\delta B\}$. Obviously, each member of $\mathcal{B}$ is symmetric. We will now prove that $\mathcal{B}$ is in fact a base for an S-uniformity $\mathcal{U}$ on X by showing that $\mathcal{B}$ is a compatible uniformity (in the sense of Th. 6.17) with δ, i.e. $A \not\delta B$ iff there exist C,D in $P(X)$ such that $C \not\delta D$ and $U_{C,D}[A] \cap B = \phi$. Let $A \not\delta B$, and suppose that $t \in U_{A,B}[A] \cap B$. Then there is an $s \in A$ such that $(s,t) \in U_{A,B}$, a contradiction as $(s,t) \in A \times B$. Hence $U_{A,B}[A] \cap B = \phi$. Conversely, suppose there exist C,D in $P(X)$ such that $C \not\delta D$ and $U_{C,D}[A] \cap B = \phi$. Then $A \subset C \cup D$; for if $x \in A - (C \cup D)$, then $U_{C,D}[x] = X$ and so $U_{C,D}[A] = X$, a contradiction as $U_{C,D}[A] \cap B = \phi$, and $B \neq \phi$. We next show that $A \subset C$ or $A \subset D$. Suppose not. Then there exist t_1, t_2 in A such that $t_1 \in C$, $t_2 \in D$, so that $U_{C,D}[t_1] = (X-D)$ and $U_{C,D}[t_2] = (X-C)$. But since $C \not\delta D$, from (P.4), $(X-C) \cup (X-D) = X$. Hence $U_{C,D}[t_1] \cup U_{C,D}[t_2] = X$ and therefore $U_{C,D}[A] = X$, which is again a contradiction as before. Thus, $A \subset C$ or $A \subset D$. Suppose $A \subset C$. Then $U_{C,D}[A] = X - D$; so $B \subset D$ and hence, by Prop. 1.14, $A \not\delta B$. The proof for the second case is similar. Hence, from Th. 6.17, $\mathcal{B}$ satisfies S-(i), S-(ii) and S-(iii). Consequently, by Th. 6.30, if we set $\mathcal{U}_1(\delta) = \{U : U = U^{-1}$ and $U \supset B$ for some B in $\mathcal{B}\}$, then $\mathcal{U}_1(\delta)$ is an S-uniformity on X. Clearly, $\delta(\mathcal{U}_1(\delta)) = \delta$, and hence the theorem is proved.

6.32 Definition: Given an S-space (X,δ), the class of all S-uniformities $\mathcal{U}$ on X such that $\delta(\mathcal{U}) = \delta$ is called a proximity class of

S-uniformities on X, denoted by $\pi(\delta)$.

6.33 Theorem : Let (X,δ) be an S-space. If $\mathcal{U} \in \pi(\delta)$, then

(a) $A \,\delta\, B$ iff for every $U \in \mathcal{U}$, $(A \times B) \cap U \neq \emptyset$.

(b) $A << B$ iff there exists an U in $\mathcal{U}$ such that $U[A] \subset B$.

Proof : (a) Let $\mathcal{U} \in \pi(\delta)$. Suppose $(A \times B) \cap U \neq \emptyset$ for every U in $\mathcal{U}$. Then clearly $U[A] \cap B \neq \emptyset$ for each U in $\mathcal{U}$ and hence $A \,\delta\, B$. Conversely, let $A \,\delta\, B$ and $U \in \mathcal{U}$. Then, as $\mathcal{U} \in \pi(\delta)$, there is a $b \in U[A] \cap B$. This means that there is an a in A such that $(a,b) \in U$. Thus $(A \times B) \cap U \neq \emptyset$.

(b) Let $\mathcal{U} \in \pi(\delta)$ and $A,B \in P(X)$. Then $A << B$ iff $A \not\delta (X-B)$ iff there is an U in $\mathcal{U}$ such that $U[A] \cap (X-B) = \emptyset$ iff there is an U in $\mathcal{U}$ such that $U[A] \subset B$.

6.34 Theorem : Let (X,δ) be an S-space. Then $\mathcal{U}_1(\delta)$, as constructed in Th. 6.31, is the least element of $\pi(\delta)$ (when $\pi(\delta)$ is partially ordered by inclusion).

Proof : Let $\mathcal{U} \in \pi(\delta)$ and $U_{A,B} \in \mathcal{U}_1(\delta)$. In order to prove that $\mathcal{U}_1(\delta)$ is the least element of $\pi(\delta)$, we must show that $U_{A,B} \in \mathcal{U}$. Since $A \not\delta B$, by Th. 6.33(a), there is a V in $\mathcal{U}$ such that $(A \times B) \cap V = \emptyset$. As $V = V^{-1}$, this means that $(B \times A) \cap V = \emptyset$. Hence, by the definition of $U_{A,B}$, $V \subset U_{A,B}$. But $\mathcal{U}$ is an S-uniformity on X and consequently, by S-(iv'), $U_{A,B} \in \mathcal{U}$, i.e. $\mathcal{U}_1(\delta) \subset \mathcal{U}$.

6.35 Theorem : Let (X,δ) be an S-space. Then the union $\mathcal{B}$ of an arbitrary family of members of $\pi(\delta)$ is a base for an S-uniformity in $\pi(\delta)$.

S-uniformities on X, denoted by $\pi(\delta)$.

6.33 Theorem : Let (X,δ) be an S-space. If $\mathcal{U} \in \pi(\delta)$, then

(a) $A \,\delta\, B$ iff for every $U \in \mathcal{U}$, $(A \times B) \cap U \neq \emptyset$.

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Proof : (a) Let $\mathcal{U} \in \pi(\delta)$. Suppose $(A \times B) \cap U \neq \emptyset$ for every U in $\mathcal{U}$. Then clearly $U[A] \cap B \neq \emptyset$ for each U in $\mathcal{U}$ and hence $A \,\delta\, B$. Conversely, let $A \,\delta\, B$ and $U \in \mathcal{U}$. Then, as $\mathcal{U} \in \pi(\delta)$, there is a $b \in U[A] \cap B$. This means that there is an a in A such that $(a,b) \in U$. Thus $(A \times B) \cap U \neq \emptyset$.

(b) Let $\mathcal{U} \in \pi(\delta)$ and $A,B \in P(X)$. Then $A << B$ iff $A \not\delta (X-B)$ iff there is an U in $\mathcal{U}$ such that $U[A] \cap (X-B) = \emptyset$ iff there is an U in $\mathcal{U}$ such that $U[A] \subset B$.

6.34 Theorem : Let (X,δ) be an S-space. Then $\mathcal{U}_1(\delta)$, as constructed in Th. 6.31, is the least element of $\pi(\delta)$ (when $\pi(\delta)$ is partially ordered by inclusion).

Proof : Let $\mathcal{U} \in \pi(\delta)$ and $U_{A,B} \in \mathcal{U}_1(\delta)$. In order to prove that $\mathcal{U}_1(\delta)$ is the least element of $\pi(\delta)$, we must show that $U_{A,B} \in \mathcal{U}$. Since $A \not\delta B$, by Th. 6.33(a), there is a V in $\mathcal{U}$ such that $(A \times B) \cap V = \emptyset$. As $V = V^{-1}$, this means that $(B \times A) \cap V = \emptyset$. Hence, by the definition of $U_{A,B}$, $V \subset U_{A,B}$. But $\mathcal{U}$ is an S-uniformity on X and consequently, by S-(iv'), $U_{A,B} \in \mathcal{U}$, i.e. $\mathcal{U}_1(\delta) \subset \mathcal{U}$.

6.35 Theorem : Let (X,δ) be an S-space. Then the union $\mathcal{B}$ of an arbitrary family of members of $\pi(\delta)$ is a base for an S-uniformity in $\pi(\delta)$.

Proof : Clearly, as every member of $\pi(\delta)$ is a symmetric family, $\mathcal{B}$ is also a symmetric family. Also, if $A, B \in P(X)$ then, by the definition of $\pi(\delta)$, $A \delta B$ iff $U[A] \cap B \neq \phi$ for each U in $\mathcal{B}$. Consequently, by applying Th. 6.17, $\mathcal{B}$ satisfies S-(i), S-(ii) and S-(iii). Hence, from Th. 6.30, $\mathcal{B}$ is a base for an S-uniformity on X which is evidently in $\pi(\delta)$.

6.36 Corollary : If (X, δ) is an S-space then $\pi(\delta)$ has a greatest element, namely the union of all the members of $\pi(\delta)$.

Proof : This is an immediate consequence of Th. 6.35.

CHAPTER 7

MORE ON SEPERATION SPACES.

Having obtained some of the basic properties of S-spaces in the previous chapter, our present objective will be to generalize the theorem of Smirnov (see Th. 2.20). to S-spaces and thereby deduce some further extension theorems when the spaces under consideration are assigned the S-proximities. Our first principal result in this direction will be to prove that every abstract S-proximity is a subspace proximity of the S-proximity δ_0' as defined in Proposition 6.3 (This generalizes the known results of Smirnov [31] and Lodato [19]). While proving this result, we shall make use of the notion of a band, a concept that corresponds to a cluster and a bunch in EF-spaces and LO-spaces respectively. A necessary and sufficient condition for an S-space to be compact will also be found. Next, we will obtain a generalization of the Smirnov Theorem for S-spaces, which will be analogous to the theorem 2.21. Finally, with the help of the above mentioned result, the theorems of Taimanov [37], McDowell [21] as well as one of our theorems, namely Theorem 3.11, will be generalized still further.

7.1 <u>Definition</u> : A non-void family σ of subsets of a p-space (X,δ) is a <u>band</u> over (X,δ) iff σ satisfies the following set of axioms :

(B*.1) : $\overline{A}\ \delta\ \overline{B}$ for every A,B in σ.

(B*.2) : $(A \cup B)\ \delta\ C$ implies $A\ \delta\ C$ or $B\ \delta\ C$.

(B*.3) : $A \in \sigma$ and $a\ \delta\ B$ for every a in A implies $B \in \sigma$.

7.2 Remark : It is not difficult to see that the axioms (B*.2) and (B*.3) are together equivalent to the two axioms (B.2) and (B.3) of 1.23.

7.3 Remark : Clearly every bunch (1.23) is a band, and the converse holds in LO-spaces. However, it need not hold in an S-space. For example, in the S-space $(\mathbb{R},\delta_0')$, $\sigma_1 = \{A \in P(\mathbb{R}) : \{1\}\ \delta_0'\ A\}$ is a band but not a bunch. This can be seen from the fact that (0,1), (1,2) are both in σ_1 but $(0,1)\ \not\delta_0'\ (1,2)$.

7.4 Proposition: If (X,δ) is an S-space, then the following hold.

(1) If σ is a band, $A \in \sigma$ and $A \subset B$, then $B \in \sigma$. Hence, in particular, $X \in \sigma$.

(2) For $A \in P(X)$ and σ a band over (X,δ), either $A \in \sigma$ or $(X-A) \in \sigma$.

(3) For $x \in X$, $\sigma_x = \{A \in P(X) : x\ \delta\ A\}$ is a band, called the point band over (X,δ).

(4) If σ is a band and $\{x\} \in \sigma$ then $\sigma = \sigma_x$.

Proof : (1) σ a band over (X,δ), $A \in \sigma$ and $A \subset B$ implies $A \in \sigma$ and $a\ \delta\ B$ for each a in A. Hence, from (B*.3), $B \in \sigma$. Thus in particular $X \in \sigma$.

(2) If σ is a band over (X,δ) then as, from (1) $X = A \cup (X-A) \in \sigma$ for each $A \in P(X)$, by (B*.2), either $A \in \sigma$ or $(X-A) \in \sigma$ for each $A \in P(X)$.

(3) To see that σ_x is a band we check (B*.1), (B*.2), and (B*.3).

(i) $A, B \in \sigma_x$ implies $x \delta A$ and $x \delta B$ implies that $x \in \overline{A} \cap \overline{B}$ (closures taken in $\tau(\delta)$) so that, by (P.4), $\overline{A} \delta \overline{B}$.

(ii) $(A \cup B) \in \sigma_x$ implies $x \delta (A \cup B)$ implies $x\delta A$ or $x \delta B$ (from (P.2)) implies $A \in \sigma_x$ or $B \in \sigma_x$.

(iii) $A \in \sigma_x$, $a \delta B$ for each a in A implies $x \delta A$ and $a \delta B$ for each a in A. Hence from P.6'', $x \delta B$ so that $B \in \sigma_x$.

Thus (B*.1), (B*.2) and (B*.3) are verified, and hence σ_x is a band.

(4) Let σ be a band and $\{x\} \in \sigma$. Then $A \in \sigma$ implies, from (B*.1), that $\overline{A} \delta \overline{x}$ i.e. $\overline{A} \delta x$ (as X is T_1). From (P.6''), this means that $A \delta x$ and hence $A \in \sigma_x$. Therefore $\sigma \subset \sigma_x$. Conversely, if $B \in \sigma_x$ then $B \delta x$, and as $\{x\} \in \sigma$, from (B*.3) it follows that $B \in \sigma$. Consequently $\sigma = \sigma_x$.

7.5 <u>Remark</u> : As a consequence of Prop. 7.4(4) it follows that if σ is a band over (X,δ) and $\sigma \supset \sigma_x$, then $\sigma = \sigma_x$. Also, we note that if $x \neq y$, x, y in X, then $\sigma_x \neq \sigma_y$. This is due to the fact that $x \neq y$ implies, from (P.5) that $x \not\delta y$; but X is T_1 and hence we get that $\overline{x} \not\delta \overline{y}$ so that both x and y cannot belong to the same band, which means $\sigma_x \neq \sigma_y$.

7.6 <u>Proposition:</u> Let (X,δ) be an S-space and $\underset{\sim}{L}$ a closed ultrafilter in $(X,\tau(\delta))$. Then $b(\underset{\sim}{L}) = \{E \in P(X) : \overline{E} \in \underset{\sim}{L}\}$ is a band over (X,δ), called the <u>band generated by</u> $\underset{\sim}{L}$.

Proof : We must again check that $b(\underset{\sim}{L})$ satisfies (B*.1), (B*.2) and (B*.3).

(i) Let $A, B \in b(\underset{\sim}{L})$. Then $\bar{A}, \bar{B} \in \underset{\sim}{L}$ which is a closed ultrafilter and hence $\bar{A} \cap \bar{B} \neq \emptyset$. From (P.4) we get $\bar{A}\, \delta\, \bar{B}$ i.e. (B*.1) holds.

(ii) $(A \cup B) \in b(\underset{\sim}{L})$ implies that $\overline{(A \cup B)} \in \underset{\sim}{L}$ implies $\bar{A} \cup \bar{B} \in \underset{\sim}{L}$ implies $\bar{A} \in \underset{\sim}{L}$ or $\bar{B} \in \underset{\sim}{L}$ (as $\underset{\sim}{L}$ is a closed ultrafilter). Hence either $A \in b(\underset{\sim}{L})$ or $B \in b(\underset{\sim}{L})$, thus verifying (B*.2).

(iii) Let $A \in b(\underset{\sim}{L})$ and $a\, \delta\, B$ for each a in A. Then $\bar{A} \in \underset{\sim}{L}$ and $A \subset \bar{A} \subset \bar{B}$. Hence, as $\underset{\sim}{L}$ is a closed ultrafilter, $\bar{B} \in \underset{\sim}{L}$. Consequently $B \in b(\underset{\sim}{L})$ and so (B*.3) holds.

We will now obtain the necessary and sufficient condition for an S-space to be compact.

7.7 Theorem : An S-space (X,δ) is compact iff every band $b(\underset{\sim}{L})$ generated by a closed ultrafilter $\underset{\sim}{L}$ on X is a point band.

Proof : The result follows from the observation that $b(\underset{\sim}{L})$ is a point band σ_{x_0} for some x_0 in X iff $\{x_0\} \in b(\underset{\sim}{L})$ iff $\{x_0\} \in \underset{\sim}{L}$ (because $\{x_0\}$ is closed in X) iff $\underset{\sim}{L}$ converges to x_0. Hence every $b(\underset{\sim}{L})$ over (X,δ) is a point band iff every closed ultrafilter $\underset{\sim}{L}$ in X is convergent iff X is compact.

7.8 Remark : It is of interest to note that, in a separated p-space (X,δ), the following hold :

(i) Every band is contained in a maximal band, (i.e. a band which is not properly contained in any other band).

(ii) If (X,δ) is an EF-space, then a non-void family σ of subsets of X is a cluster iff it is a maximal band and every band in (X,δ) is then contained in a unique cluster.

The proof of (i) is exactly similar to that of Prop. 1.25; while (ii) follows immediately from Prop. 2.2 and Prop. 2.3 respectively by recalling that in an EF-space, every band is a bunch (see Remark 7.3).

7.9 Lemma : Let (X,δ) be an S-space and let Σ'_X be the family of all bands over (X,δ). If $\Sigma' \subset \Sigma'_X$, then the 'Cl' operator defined on $P(\Sigma')$ as in Def. 2.4 is a Kuratowski closure operator.

Proof : We must check the four Kuratowski closure axioms.

(i) $Cl(\phi) = \phi$: Suppose $\sigma \in Cl(\phi)$. Then, as every subset of X absorbs ϕ (trivially), it follows that every subset of X must be in σ. In particular, $\phi \in \sigma$ and $X \in \sigma$. From (B*-1), $\bar{\phi}\ \delta\ \bar{X}$ i.e. $\phi\ \delta\ X$ which contradicts (P.3). Hence $Cl(\phi) = \phi$.

(ii) $\underset{\sim}{A} \subset Cl(\underset{\sim}{A})$, for $\underset{\sim}{A}$ in $P(\Sigma')$: Let $\sigma \in \underset{\sim}{A}$. Then every $E \in P(X)$ which absorbs $\underset{\sim}{A}$ is in σ and hence $\sigma \in Cl(\underset{\sim}{A})$.

(iii) $Cl(\underset{\sim}{A}) = Cl(Cl(\underset{\sim}{A}))$: Let $\sigma \in Cl(Cl(\underset{\sim}{A}))$ and E absorbs $\underset{\sim}{A}$. Clearly, E also absorbs $Cl(\underset{\sim}{A})$ (by the definition of $Cl(\underset{\sim}{A})$) and hence $E \in \sigma$. So $\sigma \in Cl(\underset{\sim}{A})$. The reverse inclusion follows from (ii).

(iv) $Cl(\underset{\sim}{A} \cup \underset{\sim}{B}) = Cl(\underset{\sim}{A}) \cup Cl(\underset{\sim}{B})$: Let $\sigma \in Cl(\underset{\sim}{A} \cup \underset{\sim}{B})$ and let A absorb $\underset{\sim}{A}$, B absorb $\underset{\sim}{B}$. Then, from Prop. 7.4, $A \cup B$ absorbs $\underset{\sim}{A} \cup \underset{\sim}{B}$, and hence, from (B*-2), either $A \in \sigma$ or $B \in \sigma$, i.e. $\sigma \in Cl(\underset{\sim}{A}) \cup Cl(\underset{\sim}{B})$. Conversely, if $\sigma \in Cl(\underset{\sim}{A}) \cup Cl(\underset{\sim}{B})$, then either $\sigma \in Cl(\underset{\sim}{A})$ or $\sigma \in Cl(\underset{\sim}{B})$. If E absorbs $\underset{\sim}{A} \cup \underset{\sim}{B}$, then E also absorbs each of $\underset{\sim}{A}$ and $\underset{\sim}{B}$ and hence $E \in \sigma$. Thus $\sigma \in Cl(\underset{\sim}{A} \cup \underset{\sim}{B})$.

7.10 Definition : The topology τ_A induced on Σ' by the above Kuratowski closure operator 'Cl' is called the Absorption topology (or the A-topology) on Σ'.

7.11 Remark : Note that for $A \in P(X)$, $Cl(A) = \{\sigma \in \Sigma' : A \in \sigma\}$.

7.12 Lemma : The A-topology τ_A on $\Sigma' \subset \Sigma'_X$ is :

(i) T_1 iff $\sigma_1, \sigma_2 \in \Sigma'$, $\sigma_1 \neq \sigma_2$, implies that $\sigma_1 \not\subset \sigma_2$ and $\sigma_2 \not\subset \sigma_1$.

(ii) Hausdorff if either $A \in \sigma$ or $B \in \sigma'$, $\sigma, \sigma' \in \Sigma'$, for all A, B in $P(X)$ such that $A \cup B = X$, implies that $\sigma = \sigma'$.

Proof : (i) follows as in Lemma 2.7. For (ii) we first note that if $\sigma, \sigma' \in \Sigma'$ and $\sigma \neq \sigma'$, then there exist A, B in $P(X)$ such that $A \cup B = X$ but $A \notin \sigma$ and $B \notin \sigma'$. This implies that $\sigma \notin Cl(A)$ and $\sigma' \notin Cl(B)$ (see Remark 7.11). Also, as $A \cup B = X$, $Cl(A) \cup Cl(B) = Cl(A \cup B) = \{\sigma \in \Sigma' : X \in \sigma\} = \Sigma'$ and hence $\sigma \in (\Sigma' - Cl(A))$, $\sigma' \in (\Sigma' - Cl(B))$ and $(\Sigma' - Cl(A)) \cap (\Sigma' - Cl(B)) = \phi$. Thus τ_A is Hausdorff.

7.13 Lemma : Let Y be a T_1-space and $f : X \to Y$ be a one-to-one map. Let δ_1 be the S-proximity on X as defined in Th. 6.15 corresponding to the S-proximity $\delta_2 = \delta_0'$ on Y. Then, for each y in $f(X)$, $\sigma^{(y)} = \{A \in P(X) : y \in f(\overline{A})\}$ is a band over (X, δ_1). In fact, $\sigma^{(y)} = \sigma_x$, where $x = f^{-1}(y)$.

Proof : Follows from the fact that $A \in \sigma^{(y)}$ iff $y \in f(\overline{A})$ iff $x \in \overline{A}$ iff $x \delta A$ iff $A \in \sigma_x$.

We now come to the problem of embedding an S-space (X, δ) proximally in a Hausdorff space Y with the S-proximity δ_0'. (cf. 2.19).

7.14 Theorem : Let X be a non-void set and δ a binary relation on $P(X)$. For each $A \in P(X)$ let $A^* = \{x \in X : x \,\delta\, A\}$. Then the following are equivalent :

(A) : There exists a Hausdorff space Y and a one-to-one map $f : X \to Y$ such that

(i) $f(X)^- = Y$

(ii) $f(A^*) = f(A)^- \cap f(X)$

(iii) $A \,\delta\, B$ iff $f(A) \,\delta_0'\, f(B)$ in Y. (δ_0' is the S-proximity)

(B) : δ is an S-proximity on X satisfying the following : there exists a family Σ' of bands over X such that :

(i) $A \,\delta\, B$ implies there exists a σ in Σ' such that $A, B \in \sigma$ and $\sigma = \sigma_x$ for some $x \in (A \cup B)$.

(ii) If $\sigma, \sigma' \in \Sigma'$ and if either $A \in \sigma$ or $B \in \sigma'$ for all $A, B \in P(X)$ such that $A \cup B = X$, then $\sigma = \sigma'$.

Proof : (A) $\Rightarrow$ (B) : Th. 6.15 and condition (A) (iii) show that δ is an S-proximity on X. Again, as f is p-continuous, it is also continuous and consequently X, being the continuous pre-image of a Hausdorff space, is Hausdorff. We note that $A^* = A^-$. Let $\Sigma' = \{\sigma^{(y)} : y \in f(X)\}$. Then from Lemma 7.13, Σ' is a family of bands over (X,δ). We now show that conditions (B)(i), (B)(ii) are satisfied. $A \,\delta\, B$ implies $(f(A) \cap \overline{f(B)}) \cup (\overline{f(A)} \cap f(B)) \neq \phi$. Suppose $y \in f(A) \cap \overline{f(B)}$. Then $\sigma^{(y)} \in \Sigma'$ and clearly $A \in \sigma^{(y)}$. From (A)(ii), $y \in f(\overline{B})$, i.e. $B \in \sigma^{(y)}$. By Lemma 7.13, $\sigma^{(y)} = \sigma_x$ for some x in A. Similarly, if $y \in \overline{f(A)} \cap f(B)$, then $\sigma^{(y)} = \sigma_x$ for x in B. Hence (B)(i) holds. To prove (B)(ii), suppose $\sigma^{(y_1)}, \sigma^{(y_2)} \in \Sigma'$, $y_1 \neq y_2$.

Since Y is Hausdorff, there exist disjoint nbhds. V_1, V_2 of y_1, y_2 respectively. Set $A = f^{-1}(f(X) - V_2)$, $B = f^{-1}(f(X) - V_1)$. Then $A \notin \sigma^{(y_2)}$, $B \notin \sigma^{(y_1)}$ and $f(A \cup B) = f(X)$ implies $A \cup B = X$.

(B) $\Rightarrow$ (A) : As for $x \in X$, $x \cap x \neq \emptyset$, from (P.4) $x \, \delta \, x$ and hence, by (B)(i), there exists a σ in Σ' such that $\{x\} \in \sigma$. By Prop. 7.4(4), $\sigma = \sigma_x$. Let $Y = \Sigma'$ and define $f : X \to Y$ by $f(x) = \sigma_x \in \Sigma'$. Clearly f is one-to-one. Assign Y the A-topology τ_A as given in Def. 7.10. Then, because of B(ii), from Lemma 7.12 (ii) it follows that τ_A is Hausdorff. From Remark 7.11, it is also clear that $f(X)^- = \{\sigma \in \Sigma' : X \in \sigma\} = Y$, i.e. (A)(i) holds. Again, since $\sigma_x \in f(A^*)$ iff $x \, \delta \, A$ iff $A \in \sigma_x$ iff $\sigma_x \in \overline{f(A)} \cap f(X)$, (A) (ii) is also valid. Finally, we verify (A) (iii). Let $A \, \delta \, B$. Then, there is a σ in Σ' such that $A, B \in \sigma = \sigma_x$ for some x in $(A \cup B)$. Suppose $x \in A$. Then $\sigma = \sigma_x \in f(A)$ and $B \in \sigma$ implies $\sigma \in \overline{f(B)}$, i.e. $\sigma \in f(A) \cap \overline{f(B)}$ and consequently $f(A) \, \delta_0' \, f(B)$. The proof when $x \in B$ is similar. Conversely, if $y \in (f(A) \cap \overline{f(B)}) \cup (\overline{f(A)} \cap f(B))$ then $y = \sigma_x$ for some x in $(A \cup B)$. If $x \in A$, then $\sigma_x \in \overline{f(B)}$ implies $B \in \sigma_x$ and hence $x \, \delta \, B$, i.e. $x \in A \cap \overline{B}$. But this means that $A \, \delta_0' \, B$, and hence, in view of Th. 6.7, $A \, \delta \, B$. The case when $x \in B$ can be proved in a similar fashion. Hence (A) (iii) is also verified.

The subsequent results 7.15 to 7.17 correspond to 2.9, 2.11 and 2.12 respectively.

7.15 <u>Lemma</u> :Let (X,δ) be an S-space and let $\Phi = \Phi_X : X \to \Sigma'_X$ be defined by $\Phi(x) = \sigma_x$, the point band. Then Φ is a homeomorphism of X onto a dense subspace of Σ'_X.

Proof : The result follows from a reasoning which is exactly similar to that in the proof of Th. 2.9.

7.16 The Fundamental Extension Theorem : Let (X,δ_1) and (Y,δ_2) be S-spaces and let $f : X \to Y$ be p-continuous. Then there exists an associated function $f_\Sigma : \Sigma'_X \to \Sigma'_Y$ defined by $f_{\Sigma'}(\sigma)=\{E \in P(Y) : f^{-1}(E^-) \in \sigma\}$. The map $f_{\Sigma'} : (\Sigma'_X,\tau_A) \to (\Sigma'_Y,\tau_A)$ is continuous and for $x \in X$, $f_{\Sigma'}(\sigma_x) = \sigma_{f(x)}$. Hence, identifying X with $\Phi_X(X)$ and Y with $\Phi_Y(Y)$, $f_{\Sigma'}$ is a continuous extension of f.

Proof : We first show that if $\sigma \in \Sigma'_X$, then $f_{\Sigma'}(\sigma) \in \Sigma'_Y$

(i) $A,B \in f_{\Sigma'}(\sigma)$ implies $f^{-1}(A^-)$, $f(B^-) \in \sigma$ implies $f^{-1}(A^-)\ \delta_1\ f^{-1}(B^-)$ (from (B*-1)). Since f is p-continuous, this means that $\bar{A}\ \delta_2\ \bar{B}$ and hence (B*.1) is verified for $f_{\Sigma'}(\sigma)$.

(ii) $(A \cup B) \in f_{\Sigma'}(\sigma)$ implies $f^{-1}(\overline{A \cup B}) \in \sigma$ implies $f^{-1}(\bar{A}) \cup f^{-1}(\bar{B}) \in \sigma$ implies $f^{-1}(A^-) \in \sigma$ or $f^{-1}(B^-) \in \sigma$ (from (B*-2)) so that either $A \in f_{\Sigma'}(\sigma)$ or $B \in f_{\Sigma'}(\sigma)$. Consequently (B*.2) follows .

(iii) $A \in f_{\Sigma'}(\sigma)$, $a\ \delta_2\ B$ for each a in A implies $f^{-1}(\bar{A}) \in \sigma$ and $A \subset \bar{A} \subset \bar{B}$. From Prop. 7.4(1), $f^{-1}(\bar{B}) \in \sigma$ so that $B \in f_{\Sigma'}(\sigma)$, thereby verifying (B*.3). Thus, $f_{\Sigma'}(\sigma) \in \Sigma'_Y$. The proof that $f_{\Sigma'} : (\Sigma'_X,\tau_A) \to (\Sigma'_Y , \tau_A)$ is continuous and that for x in X, $f_{\Sigma'}(\sigma_x) = \sigma_{f(x)}$ is the same as in Th. 2.11.

7.17 Theorem : Let X be a dense subspace of an S-space (T,δ_1). Then, if τ_A is the A-topology on Σ'_X , the map $\Psi = \Psi_T : (T,\tau(\delta)) \to (\Sigma'_X,\tau_A)$ defined by, for $x \in T$, $\Psi(x) = \sigma^x = \{E \in P(X) : x\ \sigma_1\ E\}$ is continuous.

If $x \in X$, then $\Psi(x) = \sigma_x$, the point band, i.e. $\Psi/X = \overline{\Phi}_X$. Further, if T is T_3, then Ψ is a homeomorphism of T in Σ'_X.

<u>Proof</u> : We need only check that $\sigma^x \in \Sigma'_X$, as the rest of the proof is identical with that of Th. 2.12. For this, we note the following :

(i) $A, B \in \sigma^x$ implies $x \delta A$ and $x \delta B$, i.e. $\overline{A} \cap \overline{B} \neq \phi$, so that, by (P.4), $\overline{A} \delta \overline{B}$.

(ii) That $(A \cup B) \in \sigma^x$ implies $A \in \sigma^x$ or $B \in \sigma^x$ is obvious.

(iii) Let $A \in \sigma^x$ and $a \delta B$ for each a in A. Then $x \in \overline{A} \subset \overline{B}$ so that $x \delta B$, i.e. $B \in \sigma^x$. This shows that σ^x satisfies (B*.1), (B*.2) and (B*.3) and hence $\sigma^x \in \Sigma'_X$.

By a reasoning exactly similar to that given in the proof of Th. 2.18, we get the following corresponding result for the S-spaces.

7.18 <u>Theorem</u> : Let (X,δ) be an S-space and let X^{**} be the family of all maximal bands in X with the A-topology τ_A. Then X^{**} is a compact T_1-space containing a dense homeomorphic copy of X.

To obtain a generalization of the Smirnov Theorem (cf. 2.20, 2.21) for the S-spaces, we first observe that the method of the proof of Corollary 2.10 also gives us an identical result when Σ and Σ_X are replaced by Σ' and Σ'_X respectively. Hence, as in Th. 2.21, we get the generalization of the Smirnov theorem which reads as follows :

7.19 <u>Theorem</u> : Let (X,δ) be an S-space such that if $A \delta B$ then there exists a band over (X,δ) which contains A and B. Then :

(i) there exists a compact T_1-space $\underset{\sim}{X}$ (the space of all maximal bands over (X,δ) with the A-topology τ_A) containing a dense homeomorphic copy of X.

(ii) $A \,\delta\, B$ iff $(\mathrm{Cl}\,(\Phi(A)) \cap \Phi(B)) \cup (\Phi(A) \cap \mathrm{Cl}\,\Phi(B)) \neq \phi$ in $\underset{\sim}{X}$.

(iii) if (Y,δ') is another S-space and if $f : (X,\delta) \to (Y,\delta')$ is p-continuous, then f has a continuous extension $f_{\Sigma'} : \underset{\sim}{X} \to (\Sigma'_Y, \tau_A)$.

We will now prove a general theorem giving necessary and sufficient conditions for continuous extensions of continuous functions from dense subspaces to exist. This result will then be used to get improvements of the results of Taimanov (Th. 3.2) and McDowell (Th. 3.6).

7.20 Theorem : Let X be a dense subspace of an S-space (T,δ'_o), (Y,δ) be a separated EF-space, and let $\underset{\sim}{Y}$ be the Smirnov compactification of (Y,δ). Then a continuous function $f : X \to Y$ has a continuous extension $\overline{f} : T \to \underset{\sim}{Y}$ iff f is p-continuous.

Proof : Necessity is evident in view of Prop. 6.14. To prove the sufficiency, we note that the map $\theta = \theta_Y : \Sigma_Y \to \underset{\sim}{Y}$, given by $\theta(\sigma) = \sigma_\theta$, the unique cluster containing $\sigma \in \Sigma_Y$, is continuous (Th. 2.13). Hence the result follows from Th. 7.17, Th. 7.16, by setting $\overline{f} = \theta_Y \circ f_{\Sigma'} \circ \Psi_T$ (We note here that, as Y is a separated EF-space, in view of Remark 7.3, $\Sigma'_Y = \Sigma_Y$).

If, in the above theorem, Y is compact Hausdorff, then the map $\theta : \underset{\sim}{Y} \to Y$ which assigns to each cluster in $\underset{\sim}{Y}$ its limit in Y, is continuous and so we may consider the extension $\overline{f}$ as a map from T into Y. Hence, we have the following theorem, the sufficiency part being an improvement of Taimanov's Theorem (Th. 3.2).

7.21 Theorem : Let X be a dense subspace of a T_1-space T and let Y be a compact Hausdorff space. Then a continuous map $f : X \to Y$ has a continuous extension $\overline{f} : T \to Y$ iff for every pair of disjoint closed sets

F_1, F_2 in Y, $(f^{-1}(F_1) \cap Cl_T\, f^{-1}(F_2)) \cup (Cl_T\, f^{-1}(F_1) \cap f^{-1}(F_2)) = \phi$.

We next prove an improved version of McDowell's Theorem (see Th.3.6).

7.22 Theorem : Let X be a dense subspace of a T_1-space E, and let functionally separated sets A, B in X satisfy the condition :

$$(A \cap Cl_E(B)) \cup (Cl_E(A) \cap B) = \phi .$$

Let Y be a Tychonoff space. Then every continuous map $f : X \to Y$ can be extended to a continuous mapping $\overline{f} : E_Y \to Y$ defined by, for $x \in E_Y$, $\overline{f}(x) = y$ iff $x \in E_y$. Moreover, E_Y is the largest subspace of E to which f has a continuous extension.

Proof : The result follows as in proof of Th. 3.8, when the δ_o LO-proximity on E is replaced by the δ_o' S-proximity on E.

7.23 Lemma : Let X be a dense subspace of an S-space (T, δ_o') and let (Y, δ') be a T_3 S-space. If a continuous map $f : X \to Y$ has a continuous extension $\overline{f} : T \to Y$, then :

(i) f is p-continuous.

(ii) for each $t \in T$, the family $f_{\Sigma'}(\sigma^t) = \{A \in P(Y) : t \in Cl_T\, f^{-1}(\overline{A})\}$ converges to some unique y^t in Y.

Proof : (i) is a direct consequence of Prop. 6.14, while the proof for (ii) is similar to the proof for the 'necessary' part of Lemma 3.9.

With the help of the above lemma, we shall prove the final results of this work.

7.23 Theorem : Let X be a dense subspace of a T_1 space T and let Y be a locally compact Hausdorff space. Then a continuous map $f : X \to Y$ has a continuous extension $\overline{f} : T \to Y$ iff :

(i) for every pair of disjoint closed sets F_1, F_2 in Y, at least one of which is compact,

$(f^{-1}(F_1) \cap Cl_T\, f^{-1}(F_2)) \cup (Cl_T\, f^{-1}(F_1) \cap f^{-1}(F_2)) = \phi$.

(ii) for each $t \in T$, there exists a compact subset C_t of Y such that $t \in Cl_T\, f^{-1}(C_t)$.

Proof : By assigning the δ_Y EF-proximity as defined in the proof of Th. 3.11 and by letting X have the induced subspace proximity δ_X from the S-space (T, δ'_o), if the extension exists, then the necessity of (i) and (ii) follows from Lemma 7.22 and the fact that Y is locally compact Hausdorff. Conversely, as condition (i) implies that f is p-continuous, from Th. 7.20 f has a continuous extension $\overline{f} : T \to Y \cup \{\infty\}$. To show that $\overline{f}(T) \subset Y$, as in the proof of Th. 3.11, for $t \in T$, we obtain a set G such that $\infty \in G \subset \overline{G} \subset (Y - C_t)$, where $C_t \in f_{\Sigma'}(\sigma^t)$. Clearly, $G \not\delta_Y C_t$, and as $f_{\Sigma'}(\sigma^t)$ is a band over (Y, δ_Y) which is a separated EF-space, $f_{\Sigma'}(\sigma^t)$ is also a bunch over (Y, δ_Y) and thus, as before, $G \notin f_{\Sigma'}(\sigma^t)$. Hence, $\overline{f}(T) \subset Y$.

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INDEX OF NOTATIONS

$P(X)$	Power set of X .
iff	if and only if .
i.e.	that is .
$A \subset B$	A is a subset of B .
ϕ	the null set .
δ	proximity relation .
$A \, \delta \, B$	$(A,B) \in \delta$.
$A \not\delta B$	$(A,B) \notin \delta$.
$\tau(\delta)$	topology induced by δ .
$\mathcal{U}(\delta)$	uniformity induced by δ .
$\delta(\mathcal{U})$	proximity induced by the uniformity $\mathcal{U}$.
$\pi(\delta)$	set of all uniformities compatible with δ .
nbhd.	neighbourhood.
$\bar{A}$ or $Cl(A)$	closure of A .
A°	interior of A.
$A << B$	B a δ-neighbourhood of A.
$\mathbb{R}$	the real line.
$\mathbb{N}$	set of natural numbers.
w.r.t.	with respect to.
f.i.p.	finite intersection property.
c.i.p.	countable intersection property.
c.p.	countably productive.
p.f.	point finite.

U^{-1}	$\{ (y,x) \in X \times X : (x,y) \in U \}$.
$U[A]$	$\{ y \in X : (x,y) \in U$ for some x in $A \}$.
Δ	diagonal in $X \times X$, i.e. $\{(x,x)\}: x \in X\}$.
δ_Y	subspace proximity on Y.
δ_F	functionally distinguishable EF-proximity.
LO-proximity δ_o	LO-proximity defined on the power set of X by $A \, \delta_o \, B$ iff $\overline{A} \cap \overline{B} \neq \phi$.
S-proximity δ_o'	S-proximity defined on the power set of X by $A \, \delta_o \, B$ iff $(A \quad \overline{B}) \quad (\overline{A} \quad B) \neq \phi$.
σ	cluster, bunch or band.
σ_x	$\{ A \in P(X) : A \, \delta \, x \}$.
$\delta_1 > \delta_2$	δ_1 is finer than δ_2 i.e. $A \, \delta_1 \, B$ implies $A \, \delta_2 \, B$.
$Z(f), Z$	zero set of the continuous function f.
$Z(X)$	family of all zero sets in X.
$W(X,\mathfrak{L}), W(\mathfrak{L})$	Wallman compactification of X corresponding to the base $\mathfrak{L}$.
$\eta(X,\mathfrak{L}), \eta(\mathfrak{L})$	Wallman $\mathfrak{L}^*$-realcompactification of X corresponding to the base $\mathfrak{L}$.
βX	Stone-Cech compactification of X.
νX	Hewitt realcompactification of X.
w	Wallman mapping.
τ_A	Absorption topology or A-topology.
Σ_X	family of all bunches over a LO-space (X,δ).
Σ'_X	family of all bands over an S-space (X,δ).
S-uniformity	separated uniformity.
SU-space	separated uniform space.
SU-topology	separated uniform topology.